BIRZEIT UNIVERSITY

Faculty of Graduate Studies and Research كلية الدراسات العليا والأبحاث

OPTIMAL WELFARE STRATEGIES FOR TWO WAGE-EARNERS WITHIN A MARKETS OF LIFE-INSURANCE AND WELFARE PROVIDERS

Prepared By
Ibtihal Idrees

Supervised By
Dr. Abdelrahim Mousa

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This Thesis was submitted in partial fulfillment of the requirements for the Master's Degree in Mathematics from the Faculty of Science at Birzeit University, Palestine.

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This Thesis was defended successfully and approved by

Committee Members:		Signature and Date
Dr. Abdelrahim Mousa	(Head of Committee)	....
Dr. Muna Abu Alhalawa	(Internal Examiner)	....
Dr. Hassan Abu Hassan	(Internal Examiner)	....

Birzeit University

2023

الإهداء

إلى أبي الذي شابهت.

إلى أمي التي رأتني بقلبها.

إلى شريكتي في كل تفاصيلي، تعاهدنا على شق الصعاب سويا وتقاسمنا قوت يومنا وأحلامنا، تشاركنا ما لا يمكن مشاركته إلا حزني فلم أستطع مشاركتها به يوما.

إلى الاشخاص الذين ساروا هذا الطريق معي.

وللوطن حصته من الإهداء.

Declaration

I certify that this Thesis, submitted for the Master degree in Mathematics to the Department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this Thesis (or any part of it) has not been submitted for a higher degree to any other university or institution.

Ibtihal Idrees Signature

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Abstract

In this Thesis, we will construct a continuous time model for a two wage-earners who are trying to find an optimal strategies concerning consumption, investment, life-insurance purchase and best welfare selection. Assuming correlated lifetimes for the two wage-earners, we consider a stochastic optimal control problem for each wage-earner before and after the first death. We assume that both wage-earners contribute in a social welfare system, have access to a financial and life-insurance markets. We use Copula model as stochastic mortality model for dependent lives, to handle the stochastic optimal control problems under consideration. For each stochastic optimal control problem, we use dynamic programming principle to derive a nonlinear second order partial differential equation, known as Hamilton-Jacobi-Bellman (HJB) equation, whose solution is the objective functional for the problem under consideration. Assuming special class of discounted constant relative risk aversion (CRRA) utilities we find an explicit solutions for possible cases with more details.

الملخص

في هذه الرسالة سنقوم ببناء نموذج زمني مستمر لإثنين من أصحاب الأجور اللذين يحاولان إيجاد الإستراتيجيات المثلى فيما يتعلق بالإستهلاك والإستثمار وشراء التأمين على الحياة واختيار أفضل رعاية اجتماعية. على فرض وجود علاقة بين فترات الحياة لكلا أصحاب الأجور، فإننا نقدم مشكلة التحكم العشوائي لكل صاحب أجر قبل وبعد الوفاة الأولى. نحن نفترض أن أصحاب الأجور يساهمان في نظام الرعاية الاجتماعية، ويتمتعون بإمكانية الوصول إلى أسواق التأمين المالي والتأمين على الحياة. نحن نستخدم نموذج كوبولا، كنموذج لمعدل الوفيات العشوائي الخاص بفترات الحياة غير المستقلة، وذلك للتعامل مع مشاكل التحكم الأمثل والعشوائية المعتمدة قيد الدراسة. لكل مشكلة تحكم أمثل وعشوائية، نستخدم مبدأ البرمجة الديناميكية لإشتقاق معادلة تفاضلية جزئية غير خطية من الدرجة الثانية، والمعروفة باسم معادلة هاملتون جاكوبي بيلمان، والتي يكون حلها هو اقتران الهدف للمشكلة قيد الدراسة. بافتراض فئة خاصة من اقترانات المنفعة ذات المخاطر النسبية الثابتة نجد حلولًا مباشرة للحالات المحتملة المطروحة مع مزيد من التفاصيل.

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Symbols

$\emptyset$	Empty set
$\mathbb{N}$	Natural numbers
$\mathbb{N}_0$	Natural numbers including zero
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive Real numbers
$\mathbb{R}^n$	Space of n-dimension
$\mathcal{F}$	Sigma algebra on Ω
$\mathcal{F}_t$	Filtration
inf	Infimum
sup	Supremum
min	Minimum
max	Maximum
ODE	Ordinary differential equation
PDE	Partial differential equation
SDE	Stochastic differential equation
$\mathbb{P}$	Probability measure
HJB	Hamilton-Jacobi-Bellman
DPP	Dynamic Programming Principle
$\mathbb{E}$	Expected value
CRRA	Constant Relative Risk Aversion
$C(\cdot,\cdot)$	Copula function
$T \wedge \tau_i$	$= min\{T, \tau_i\}$
$\forall$	For all
$\exists$	There exists
$\in$	Belong to

Chapter 1

Introduction

We consider the problem of a two wage-earners having to make selections about strategies: consumption, investment, life-insurance purchase and social security system over the interval of time $[0, min\{T, \tau_i\}]$ where

- T is a fixed time in the future and representing the common retirement time of the two wage-earners.
- τ_i is a random variable and representing the death time of the wage-earner i where $i = 1, 2$.

Each of the wage earners have income at a continuous rate $I_i(\cdot), i = 1, 2$ and when the wage-earner retires or dies this income ends, whichever happens first. Furthermore, we assume that each wage-earner participates in social security with the aim of protecting his family in the future, maximize the value of his legacy in the case of premature death, or the value of his wealth at retirement date T if he lives that long.

Several optimization problem containing personal consumption and life-insurance purchase Yaari [46] in 1956, who showed that a wage-earner with an uncertain lifetime and a fixed stock of resources should purchase an annuity contract to insure against the risk. Marshall [28] presented some derivations of a multivariate exponential distribution. One of these derivations assumes the residual life is independent of age, and other derivations are based on classical models of dependent lives which called common-shock models. These models assume that the dependence of lives arises from an exogenous event that is common to each life. For example, in lifetime analysis this shock may be an accident or the onslaught of a contagious disease. Yaari work was extended by Hakansson [15] to model of firm under risk. Merton in [29, 30] focused on the optimal consumption, investment but without life-insurance. Richard in [41] generalized the work of Merton [29, 30] to include the life-insurance in a continuous-time model for an uncertain lived wage-earner.

Frees et al [11] studied the use of dependent mortality models to value type of annuity, discussed a broad class of parametric bivariate survival models using a bivariate survivorship function called a Copula. Moore and Young [32] studied possible strategies to insurance wage-earner in a continuous-time model. Ye [47] examined the intertemporal model of optimal consumption, life-insurance purchase and portfolio rules for an individual whose lifetime is uncertain in a quite complex continuous-time economy.

The work done by Luciano et al [27] used Copula and common-shock model to studied the stochastic mortality of couples. In the same year, Kraft and Steffensen [22] studied problems about consumption and insurance model in a continuous-time multi-state Markovian framework. A Markov model assumes that the probability depends only on the current time and the state occupied, that is, it is independent of the past given the present value (that is, it assumes the Markov property). Kwak et al [24] investigate an optimal portfolio, consumption and retirement decision problem in which a wage-earner can determine the discretionary stopping time as a retirement time with constant labor wage and disutility. Additionally, Bruhna and Steffensen [3] developed a continuous-time Markov model for maximization problem using power utility function of a two-person household. Ji et al [21] in (2011) studied joint life mortality risk evaluation and management in the Markovian framework.

Recently in (2011), Kwak et al [25] investigate an optimal investment, consumption and life-insurance decision problem of the family with a one breadwinner (a parent) and one dependent (a child) using hyperbolic absolute risk aversion (HARA) utility functions. Using HARA specification effectively impose the axiomatic condition that instantaneous consumption rate must be above a lower bound while the breadwinner is alive and after his or her death. Pirvu and Zhang [40] studied optimal investment, consumption and life-insurance acquisition strategies for an individual who uses an expected utility criterion with discounted Constant Relative Risk Aversion (CRRA) type preferences. CRRA is an increasing and strictly concave utility function U with the property $-c\frac{U''(c)}{U'(c)}$ is constant.

Later on, Bayraktar and Young [2] studied the optimal amount of life insurance for a household of two wage-earners. Kwak et al [23] studied the portfolio decisions of a wage-earner under inflation risks. Park and Jang [38] studied an optimal consumption and portfolio selection problem of an individual who wants to voluntarily retire someday in the future. Mousa et al [34] studied the problem faced by a wage-earner whose aim is to maximize consumption and investment within a diminishing basket of K goods in a financial market model involved of one risk-free security and an arbitrary number of risky securities.

Mousa et al [35] studied the optimal life-insurance within a market contains many life-insurance companies using dynamic programming techniques to find an explicit solution in the case of discounted CRRA utility functions in a continuous-time model for one wage-earner. Han et al [16] studied the portfolio decisions of a wage-earner under inflation risks. In (2022) Mousa et al [33] introduced an optimization problem of finding the optimal life-insurance strategies for a wage-earner with an uncertain lifetime in a financial market involved one risk- free security and one risky security whose prices evolve according to linear diffusions process. After that Hoshiea et al [18] introduced a modified version of Merton's continuous lifetime model [29, 30] where the welfare policy is being a new control variable in the problem, and find an explicit solution using CRRA utilities, in the case where the social welfare system consist of only one welfare provider.

Wei et al [45] in (2020) studied another optimization problem to determine the optimal consumption, investment and life-insurance purchase strategy for a two wage-earners with related lifetimes. Based on the work of Wei, in this Thesis, we will try to find the optimal strategies concerning consumption, investment, and life-insurance purchase for the two wage-earners within optimal welfare market. Having access to the welfare market imposes an additional control involved in the dynamics of the problem under consideration. In this work we will analyse paper [45] with more details.

Chapter 2

Preliminaries

2.1 Basics in probability and stochastic processes

In this section we will introduce some definitions and results in probability and stochastic processes, that will be used later.

Definition 2.1. *[14]* ***Probability theory***
Probability theory describes mathematical models of random phenomena, primarily from a theoretical point of view.

Definition 2.2. *[14]* ***Random experiment***
A random experiment is an experiment that can be repeated and the future outcomes cannot be exactly predicted even if the experimental situation can be fully controlled.

Definition 2.3. *[9]* ***Sample space***
The sample space Ω is any a set that lists all possible outcomes of some unknown random experiment or situation, and any $x \in \Omega$ is called sample point.

Definition 2.4. *[9]* ***Event***
The event is defined as any subset of the sample space Ω.

Definition 2.5. *[7]* σ***-algebra***
If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties

1. $\phi \in \mathcal{F}$.

2. $F \in \mathcal{F} \implies F^c \in \mathcal{F}$, *where* $F^c = \Omega \setminus F$ *is the complement of* F *in* Ω.

3. $A_1, A_2, \dots \in \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition 2.6. *[7]* ***Measurable space***

A measurable space is a pair $(\Omega, \mathcal{F})$ *for which* $\mathcal{F}$ *is a* σ*-algebra on the space* Ω.

Definition 2.7. *[7]* ***$\mathcal{F}$-measurable sets***

The subsets A *of* Ω *which belong to* $\mathcal{F}$ *are called* $\mathcal{F}$*-measurable sets.*

Definition 2.8. *[7]* ***Probability measure***

A probability measure $\mathbb{P}$ *on measurable space* $(\Omega, \mathcal{F})$ *is a function* $\mathbb{P} : \mathcal{F} \to [0, 1]$ *such that*

1. $\mathbb{P}(\emptyset) = 0$ *and* $\mathbb{P}(\Omega) = 1$.

2. *If* $A_1, A_2 \ldots \in \mathcal{F}$ *and* $\{A_i\}_{i=1}^{\infty}$ *are disjoint* ($i.e.$ $A_i \cap A_j = \emptyset$ *if* $i \neq j$) *then*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}\left(A_i\right).$$

Definition 2.9. *[7]* ***Probability space***

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a probability space for which* $\mathcal{F}$ *is a* σ*-algebra on the space* Ω *and a probability measure* $\mathbb{P}$.

Example 2.1. *[7] Let* $\Omega = \{a_1, a_2, \ldots, a_n\}$ *be a finite set, and suppose we are given numbers* $0 \leq p_i \leq 1$ *for* $i = 1, \ldots, n$, *satisfying*

$$\sum_{1}^{n} p_i = 1.$$

Assume $\mathcal{F}$ *contains all subsets of* Ω. *For each set*

$$A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\} \in \mathcal{F},$$

where $1 \leq i_1 < i_2 < \ldots i_m \leq n$, *we define*

$$P(A) = p_{i_1} + p_{i_2} + \cdots + p_{i_m}.$$

Definition 2.10. *[7]* ***Complete probability space***

A probability space is complete if $\mathcal{F}$ *contains all subsets* G *of* Ω *with* $\mathbb{P}$*-outer measure zero, i.e. with*

$$\mathbb{P}^*(G) := \inf\{\mathbb{P}(A); A \in \mathcal{F}, G \subset A\} = 0.$$

Definition 2.11. *[7] σ-algebra generated by $\mathcal{U}$*

Given any family $\mathcal{U}$ of subsets of Ω, there is a smallest σ-algebra $\mathcal{H}_{\mathcal{U}}$ containing $\mathcal{U}$, namely

$$\mathcal{H}_{\mathcal{U}} = \bigcap \{\mathcal{H}; \mathcal{H}\ \sigma - algebra\ of\ \Omega,\ \mathcal{U} \subset \mathcal{H}\},$$

we call $\mathcal{H}_{\mathcal{U}}$ the σ-algebra generated by $\mathcal{U}$.

Definition 2.12. *[7] **Borel sets***

If $\mathcal{U}$ is the collection of all open subsets of a space Ω, then $\mathcal{B} = \mathcal{H}_{\mathcal{U}}$ is called the Borel σ-algebra on Ω and the elements $B \in \mathcal{B}$ are called Borel sets.

Definition 2.13. *[7] $\mathcal{F}$-**measurable***

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space, then a function $Y : \Omega \to \mathbb{R}^n$ is called $\mathcal{F}$-measurable if

$$Y^{-1}(U) = \{\omega \in \Omega;\ Y(\omega) \in U\} \in \mathcal{F},$$

for all open sets $U \in \mathbb{R}^n$ (or, equivalently, for all Borel sets $U \subset \mathbb{R}^n$).

Definition 2.14. *[9] **Random variable***

A random variable is a function from the sample space Ω to $\mathbb{R}$.

Definition 2.15. *[7] **n-dimensional random variable***

Let probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A random variable is a function $\mathrm{X} : \Omega \to \mathbb{R}^n$ which is measurable in the sense that the inverse of a measurable Borel set B in $\mathbb{R}^n$ is in $\mathcal{F}$. That is

$$\mathrm{X}^{-1}(B) = \{\omega \in \Omega;\ \mathrm{X}(\omega) \in B\} \in \mathcal{F}.$$

Definition 2.16. *[9] **Indicator function***

If $G \in \mathcal{F}$ is any event, then we can define the indicator function of G, written 1_G, to be the random variable

$$1_G(s) = \begin{cases} 1, & if\ s \in G, \\ 0, & if\ s \notin G, \end{cases}$$

which is equal to 1 on G, and 0 on G^c.

Definition 2.17. *[8] **Almost surely***

We say the property hold almost surely if its hold except for some event with probability zero (usually written as "a.s.").

Definition 2.18. *[37] **Almost everywhere***

A property that holds for all $x \in A \setminus M$, where M is a set of measure zero, is said to hold almost everywhere.

Definition 2.19. *[9] **Conditional probability***
The conditional probability of event A given event B where $P(B) > 0$ is defined as

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Definition 2.20. *[9] **Independent***
A and B are called independent events if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Definition 2.21. *[7] **Expectation***
If

$$\int_{\Omega} \mathrm{X}(\omega) d\mathbb{P}(\omega) < \infty,$$

then the number

$$\mathbb{E}[\mathrm{X}] = \int_{\Omega} \mathrm{X}(\omega) d\mathbb{P}(\omega),$$

is called the expectation of X on Ω.

Definition 2.22. *[7] **Expectation of independent random variables***
Let random variables

$$\mathrm{X}, \mathrm{Y} : \Omega \to \mathbb{R},$$

where

$$\mathbb{E}[\mathrm{X}] < \infty,$$

and

$$\mathbb{E}[\mathrm{Y}] < \infty,$$

then X, Y are called independent if

$$\mathbb{E}[\mathrm{XY}] = \mathbb{E}[\mathrm{X}]\mathbb{E}[\mathrm{Y}].$$

Definition 2.23. *[7] **Stochastic process***
A stochastic process is a collection of random variables

$$\{\mathrm{X}_t\}_{t \in T},$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming values in $\mathbb{R}^n$. We denote by $\mathrm{X}(\cdot)$ for the stochastic process to express the randomness in X.

Definition 2.24. *[7] **Parameter space T***
The parameter space T is usually the half line $[0, \infty)$ but it may also be an interval $[a, b]$,

the non-negative integers or subsets of $\mathbb{R}^n$ *for* $n \geq 1$.

Definition 2.25. *[7]* ***Sample path***
For each $t \in T$ *fixed we have a random variable*

$$\omega \to \mathrm{X}_t(\omega)\ ; \quad \omega \in \Omega,$$

fixing $\omega \in \Omega$ *we can define the function*

$$t \to \mathrm{X}_t(\omega)\ ; \quad t \in T,$$

which is called a sample path of X_t.

Definition 2.26. *[37]* ***Measurable stochastic process***
Let a stochastic process $\{\mathrm{X}_t\}_{t\in T}$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, this process is measurable if the mapping*

$$(t, \omega) \to X_t(\omega)$$

is measurable with respect to the σ*-algebra* $\mathcal{B}(T) \times \mathcal{F}$*, where* $\mathcal{B}(T)$ *is the family of all Borel subsets of* T.

Definition 2.27. *[14]* ***Filtration***
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$*. A filtration is a sequence* $\{\mathcal{F}_t : t \geq 0\}$ *of increasing sub-*σ*-algebras of* $\mathcal{F}$ *which means that*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \cdots \subset \mathcal{F}.$$

If t *is interpreted as (discrete) time, then* $\mathcal{F}_t$ *contains the information up to time* t.

Definition 2.28. *[7]* $\mathcal{F}_t$***-adapted***
Given $\{\mathcal{F}_t\}_{t\geq 0}$ *be an increasing family of* σ*-algebras of subsets of* Ω*. A process*

$$g(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$$

is called $\mathcal{F}_t$*-adapted if for each* $t \geq 0$ *the function*

$$\omega \to g(t, \omega)$$

is $\mathcal{F}_t$*-measurable.*

Definition 2.29. *[14]* ***Natural filtration***
A sequence $\{X_n, n \geq 0\}$ *of random variables is* $\{\mathcal{F}_n\}$*-adapted if* X_n *is* $\mathcal{F}_n$*-measurable for all* n*. If* $\mathcal{F}_n = \sigma\{X_0, X_1, X_2, \ldots, X_n\}$ *then we called the sequence adapted, and* $\{\mathcal{F}_n, n \geq 0\}$ *is the natural filtration.*

Definition 2.30. *[4] **Progressively measurable***
A stochastic process $(X_t)_{t\in\mathbb{R}^+}$ *defined on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}, \mathbb{P})$ *is progressively measurable relative to filtration* $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ *if the function*

$$(s, \omega) \in [0, t] \times \Omega \to X(s, \omega),$$

is $(\mathcal{B}([0, t]) \otimes \mathcal{F}_t)$*-measurable for all* $t \in \mathbb{R}^+$.

Definition 2.31. *[4]**Predictable real-valued process***
A real-valued process is named predictable relative to filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ *, if the function*

$$\mathbb{R}^+ \times \Omega \to \mathbb{R},$$

is measurable relative to the σ*-algebra generated by the same filtration.*

Remark 2.1. *[8] Given a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and suppose* $X : \Omega \to \mathbb{R}^n$ *is a random variable. And let* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. *Then* $x \le y$ *iff* $x_i \le y_i, \ \forall i = 1, \ldots, n$.

Definition 2.32. *[8] **Distribution function***
Given a random variable X*, then the distribution function of* X *is the function*

$$F_{\mathrm{X}} : \mathbb{R}^n \to [0, 1],$$

defined as

$$F_{\mathrm{X}}(x) := \mathbb{P}(\mathrm{X} \le x) \quad , \forall x \in \mathbb{R}^n.$$

Definition 2.33. *[36] **Cumulative distribution function***
The cumulative distribution function (cdf) of any random variable X *is the function*

$$F_{\mathrm{X}} : \mathbb{R} \to [0, 1],$$

defined as

$$F_{\mathrm{X}}(x) := \mathbb{P}(\mathrm{X} \le x) \quad , \forall x \in \mathbb{R}.$$

Definition 2.34. *[14] **Joint distribution function***
The joint distribution function of random variable X *is*

$$F_{\mathrm{X_1, X_2, \ldots X_n}}(x_1, x_2, \ldots x_n) = \mathbb{P}(\mathrm{X}_1 \le x_1, \mathrm{X}_2 \le x_2, \ldots \mathrm{X}_n \le x_n),$$

for $x_k \in \mathbb{R}, k = 1, 2, \ldots, n$.

Definition 2.35. *[8]* ***Density function***

Given a random variable $\mathrm{X} : \Omega \to \mathbb{R}^n$ *and* F_{X} *its distribution function. If there exists a non-negative, integrable function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$F_{\mathrm{X}}(x) = F_{\mathrm{X_1,X_2,\ldots X_n}}(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(y_1, \ldots, y_n)\ dy_n \ldots dy_1,$$

then f *is called the density function for* X*.*

Definition 2.36. *[8]* ***Continuous of real valued random variable***

A real valued random variable X *is called continuous if its distribution function can be expressed as the integral* $F_{\mathrm{X}}(x) = \int_{-\infty}^{x} f(y)dy$*, where* f *is the density of* X*.*

Definition 2.37. *[14]* ***Marginal distribution function***

The marginal distribution function of continuous random variable X *at the point* x *is obtained by adding the values of the marginal probabilities to the left of* x

$$F_X(x) = P(X \leq x, Y < \infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(u, v)\mathrm{d}u\ \mathrm{d}v,$$

the marginal density function is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\mathrm{d}y.$$

Definition 2.38. *[8]* ***Conditional expectation***

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *and suppose* $\mathcal{U}$ *is a* σ*-algebra,* $\mathcal{U} \subseteq \mathcal{F}$*. If* $\mathrm{X} : \Omega \to \mathbb{R}^n$ *is an integrable random variable, then the conditional expectation is*

$$\mathbb{E}[\mathrm{X} \mid \mathcal{U}],$$

to be any random variable on Ω *such that*

1. $\mathbb{E}[\mathrm{X} \mid \mathcal{U}]$ *is* $\mathcal{U}$*-measurable.*
2. $\int_A \mathrm{X} d\mathbb{P} = \int_A \mathbb{E}[\mathrm{X} \mid \mathcal{U}] d\mathbb{P},\ \forall A \in \mathcal{U}.$

We can interpreted $\mathbb{E}[\mathrm{X} \mid \mathcal{U}]$ *as a* $\mathcal{U}$*-measurable random variable that is the best approximation for* X*.*

Definition 2.39. *[14]* ***Finite-dimensional product measures***

Let $(\Omega_k, \mathcal{F}_k, \mathbb{P}_k), 1 \leq k \leq n$*, be probability spaces. We introduce that*

$$\mathcal{F}_1 \times \mathcal{F}_2, \ldots, \times \mathcal{F}_n = \sigma\{F_1 \times F_2, \ldots, \times F_n : F_k \in \mathcal{F}_k, k = 1, 2, \ldots, n\}.$$

Based on above model, we can construct a product space$(\times_{k=1}^{n}\Omega_k, \times_{k=1}^{n}\mathcal{F}_k)$, with an associated probability measure P, given as

$$\mathbb{P}(A_1 \times A_2 \times \cdots \times A_n) = \prod_{k=1}^{n} \mathbb{P}_k(A_k),$$

for $A_k \in \mathcal{F}_k, 1 \leq k \leq n$.

Theorem 2.40. *[6]* ***Fubini -Tonelli Theorem***
Suppose A_1 and A_2 are σ-finite measure spaces. not necessarily complete, and if either

$$\int_{A_1} \left(\int_{A_2} f(x,y)\mathrm{d}y \right) \mathrm{d}x < \infty,$$

or

$$\int_{A_2} \left(\int_{A_1} f(x,y)\mathrm{d}y \right) \mathrm{d}x < \infty,$$

then

$$\int_{A_1 \times A_2} |f(x,y)|\mathrm{d}(x,y) < \infty,$$

and

$$\int_{A_1} \left(\int_{A_2} f(x,y)\mathrm{d}y \right) \mathrm{d}x = \int_{A_2} \left(\int_{A_1} f(x,y)\mathrm{d}x \right) \mathrm{d}y = \int_{A_1 \times A_2} f(x,y)\mathrm{d}(x,y).$$

Theorem 2.41. *[17]* ***Test for Maxima and Minima***
Assume $f(x,y)$ be a function where

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0,$$

at a point (x_0, y_0), and suppose that all second partial derivatives are continuous there. We denote by D the discriminant

$$D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2,$$

at the critical point (x_0, y_0), and conclude the following

- *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ at (x_0, y_0), then a relative minimum occurs at (x_0, y_0). In this case, $f_{yy}(x_0, y_0) > 0$ also.*
- *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ at (x_0, y_0), then a relative maximum occurs at (x_0, y_0). In this case, $f_{yy}(x_0, y_0) < 0$ also.*
- *If $D < 0$ at (x_0, y_0), there is neither a maximum nor a minimum at (x_0, y_0).*
- *If $D = 0$ at (x_0, y_0), the test fails.*

Definition 2.42. *[43] **Vector field***
A vector field is a function that assigns a vector to each point in its domain. In three-dimensional domain a vector field might have a formula like

$$\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}.$$

Note that

- *The vector field $\vec{F}$ is continuous if the component functions M, N, and P are continuous.*
- *The vector field $\vec{F}$ is differentiable if each of the component functions is differentiable.*

In two-dimensional domain a vector field might have a formula like

$$\vec{F}(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}.$$

Definition 2.43. *[37] **Brownian motion***
A one-dimensional standard Brownian motion (or Wiener process) is a continuous stochastic process $\{W_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the following properties

1. *The standard Brownian motion process begins at $t_0 = 0$ or $\mathbb{P}(W_0 = 0) = 1$.*
2. *For $0 = t_0 \leq t_1 \leq \cdots \leq t_k$, the increments displacements*

$$W(t_1), W(t_2) - W(t_1), \ldots, W(t_k) - W(t_{k-1}),$$

   are independent random variables.

3. *For $0 \leq s < t$, the increments $W_t - W_s \sim N(0, t - s)$, that is*

$$\mathbb{P}((W_t - W_s) \in \mathcal{B}) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathcal{B}} e^{-\frac{(x)^2}{2(t-s)}} dx,$$

   where $\mathcal{B}$ is a Borel subset of $\mathbb{R}$.

Remark 2.2. *1. Property (1) in Definition 2.43 means to determine the position of a Brownian particle in one dimension, we start at $t = 0$, with the initial position specified as $W_0 = 0$.*

2. *Property (2) in Definition 2.43 means the in increment*

$$W(t_1), W(t_2) - W(t_1), \ldots, W(t_{k-1}) - W(t_{k-2}),$$

occurring during the time intervals $[t_0, t_1], \ldots, [t_{k-2}, t_{k-1}]$*, respectively, do not affect the increment* $W(t_k) - W(t_{k-1})$ *that obtains during the time interval* $[t_{k-1}, t_k]$*. That is, the standard Brownian motion is assumed to be without memory. For instance, the path of a pollen particle traverses in order to get to its current position does not influence its future location.*

3. *Property (3) in Definition 2.43 indicates that*

   - $W_t - W_s$ *for* $0 \leq s < t$ *, has a zero mean (if we think of* W_t *as the height above a horizontal time-axis of pollen particles at time* t*, then a zero mean indicates that, at time* $t + 1$*, the particle's height is just as likely to increase as it is to decrease, with no upward or downward drift).*
   - *The variance* $t - s$ *of an standard Brownian motion process increases with the length of the time interval* $[s, t]$ *(the pollen particle moves away from its position at time* s*, and there is no tendency for the particle to return to that position, that is, the process lacks any propensity for position reversion).*
   - *The continuous path of Brownian motion* $t \to W(t, \omega)$*,* $t \geq 0$ *is nowhere differentiable.*

Definition 2.44. *[8]* ***n-dimensional Wiener process***
An $\mathbb{R}^n$ *the stochastic process* $\mathbf{W}(\cdot) = \left(W^1(\cdot), \ldots, W^n(\cdot)\right)$ *is an* n*-dimensional Wiener process provided*

1. *For each* $k = 1, \ldots, n, W^k(\cdot)$ *is a 1-dimensional Wiener process,*
2. *the* σ*-algebras* $\mathcal{W}^k = \mathcal{U}\left(W^k(t) \mid t \geq 0\right)$ *are independent,* $k = 1, \ldots, n$*.*

Definition 2.45. *[8]* ***White noise***
The white noise ξ *is the time derivative of the Brownian motion. Mathematically,*

$$\xi(t) = \dot{W}(t) = \frac{dW(t)}{dt}.$$

Definition 2.46. *[10]* ***Poisson process***
A stochastic process $\{X(t); t \geq 0\}$ *taking values on* $S = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$*, with the right continuous and piecewise constant trajectories is said to be a Poisson process with parameter* $\lambda > 0$ *if*

1. $X(0) = 0$*.*

2. *For* $0 = t_0 \leq t_1 \leq \cdots \leq t_n$ *, the increments*

$$X(t_0), X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1}),$$

are mutually independent random variables.

3. *For all* $s, t, h > 0$ *and* $s > t$ *the increments*

$$X(s) - X(t),$$

and

$$X(s+h) - X(t+h),$$

have the identical probability distributions (which is called stationary increments).

4. *For all* $t > 0, h \geq 0$

$$P(X(t+h) - X(t) = k) = \frac{(\lambda h)^k}{k!} e^{-\lambda h}, k \in \mathbb{N}_0.$$

Definition 2.47. *[10]* ***Counting process***
Given a stochastic process $\{N(t) : t \geq 0\}$ *the formula*

$$N(t) = \sup \{n \in \mathbb{N}_0 : \tau_n \leq t\},$$

is called a counting process corresponding to a random sequence $\{\tau_n : n \in \mathbb{N}_0\}$*.*

Definition 2.48. *[10]* ***Nonhomogeneous Poisson Process (NPP)***
Let a stochastic process $\{N(t) : t \geq 0\}$ *taking values* $S = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$*, value of which represents the number of events in a time interval* $[0, t]$*. A counting process* $\{N(t) : t \geq 0\}$ *is called a Nonhomogeneous Poisson Process (NPP) with an intensity function* $\lambda(t) \geq 0, t \geq 0$*, if*

1. $P(N(0) = 0) = 1$*.*
2. $\{N(t) : t \geq 0\}$ *is process with independent increments, the right continuous and piecewise constant trajectories.*
3. $P(N(t+h) - N(t) = k) = \frac{\left(\int_t^{t+h} \lambda(x)dx\right)^k}{k!} e^{-\int_t^{t+h} \lambda(x)dx}$*.*

Remark 2.3.
- *Poisson process represents to count the number of times events in a time interval* $[0, t]$*. For instance, number of arrivals customers to a store or number of arrivals phone calls to a call center.*

- *From the above definition it follows that the one dimensional distribution of NPP is given by the rule*

$$P(N(t)=k)=\frac{\left(\int_0^t \lambda(x)dx\right)^k}{k!}e^{-\int_0^t \lambda(x)dx}, k=0,1,2,\ldots.$$

- *The expectation and the variance of NPP are the functions given by*

$$\Lambda(t)=E[N(t)]=\int_0^t \lambda(x)dx,$$
$$\mathrm{V}(t)=V[N(t)]=\int_0^t \lambda(x)dx, t\geq 0.$$

Definition 2.49. *[44] **Continuous-time Markov chain***
A continuous-time Markov chain on a finite set S is a family of random variables $\{X(t)\}_{t\geq 0}$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that

$$\begin{aligned}&P\left(X\left(t_{n+1}\right)=j \mid X\left(t_n\right)=i, X\left(t_{n-1}\right)=i_{n-1},\ldots,X\left(t_0\right)=i_0\right)\\&=P\left(X\left(t_{n+1}\right)=j \mid X\left(t_n\right)=i\right)\\&=P_{i,j}\left(t_{n+1}-t_n\right),\ \textit{for}\quad j,i,i_{n-1},\ldots,i_0\in S\ \textit{and}\quad t_{n+1}>t_n>\ldots>t_0\geq 0.\end{aligned}$$

In other words, the future of the process is conditionally independent of the past given the present value.

2.2 Differential equations

In next three sections we will review some types of differential equations (ODE, PDE and SDE) with some examples and more details.

Definition 2.50. *[1] **Differential equation***
A differential equation is an equation, where the unknown is a function and both the function and its derivatives may appear in the equation.

2.2.1 Ordinary differential equations

In this section we will introduce definition of ordinary differential equations and method to solve ordinary differential equations with example.

Definition 2.51. *[42] **Ordinary differential equations***
An ordinary differential equations (ODE) is an equation in which the unknown quantity

is a function, and the equation involves derivatives of the unknown function with respect to only one variable.

For the fixed point $x_0 \in \mathbb{R}^n$ the ordinary differential equation may have the form

$$\begin{cases} \dot{\mathrm{x}}(t) = \mathbf{b}(\mathrm{x}(t)), & t > 0 \\ \mathbf{x}(0) = x_0, \end{cases}$$

where $\mathrm{x}(t)$ is the state of the system at time $t \geq 0$

$$\dot{\mathrm{x}}(t) = \frac{d}{dt}\mathrm{x}(t),$$

and

$$\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n,$$

is a given, smooth vector field and the solution is the trajectory

$$\mathrm{x}(\cdot) : [0, \infty) \to \mathbb{R}^n.$$

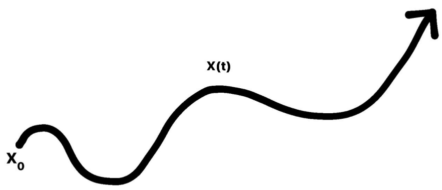

FIGURE 2.1: POSSIBLE TRAJECTORY OF ORDINARY DIFFERENTIAL EQUATIONS.

Definition 2.52. *[1]* 1^{st} ***order linear ODE***
The general first order linear ordinary differential equation defined as

$$x^{'} + p(t)x = q(t),$$

where $p(t), q(t)$ are continuous functions on an interval $I \subseteq \mathbb{R}$.

Theorem 2.53. *[1]* ***Solution of*** 1^{st} ***order linear ODE***
The general solution $x(t)$ satisfying the initial value $x(t_0) = x_0$ for any numbers $t_0 \in I$ and $x_0 \in \mathbb{R}$. Precisely,

$$x(t) = e^{-\int_{t_0}^{t} p(s)ds}\left[x_0 + \int_{t_0}^{t} e^{\int_{t_0}^{s} p(s)ds} q(s)ds\right], \quad t \in I.$$

Example 2.2. *Find the general solution of*

$$x^{'}(t) + 4tx(t) = 8t.$$

and the solution such that $x(0) = 1$.
We can use the general formula from Theorem 2.53 to find

$$\begin{aligned} x(t) &= e^{-4\int_0^t sds}\left[1 + \int_0^t e^{4\int_0^s sds} 8.sds\right] \\ &= e^{-2t^2}\left[1 + \int_0^t 8.se^{2s^2}ds\right] \\ &= e^{-2t^2}\left[1 + (2e^{2t^2} - 2)\right] \\ &= 2 - e^{-2t^2}. \end{aligned}$$

2.2.2 Partial differential equations

In this section we will introduce definition of partial differential equations with examples.

Definition 2.54. *[20]* ***Partial differential equations***
A partial differential equations (PDE) is an equation for a function which depends on more than one independent variable which involves the function, the independent variables and partial derivatives of the function. The order of an PDE is the highest derivative that appears.

- The first order PDE in two independent variables of $u(x, y)$ may have the form

$$F\left(x, y, u(x, y), u_x(x, y), u_y(x, y)\right) = 0.$$

- The second order PDE in two independent variables of $u(x, y)$ may have the form

$$F\left(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y)\right) = 0.$$

where u_x and u_y denote first-order partial derivatives with respect to x and y, respectively, u_{xx} and u_{yy} denote a second-order derivatives with respect to x and y, respectively, finally u_{xy} represent to partial derivative of u_x with respect to y.

Example 2.3. *[20] Some examples of PDEs*

1. *The transport PDE*

$$u_x + u_y = 0.$$

2. The shock wave PDE

$$u_x + uu_y = 0.$$

3. The laplace's PDE

$$u_{xx} + u_{yy} = 0.$$

2.2.3 Stochastic differential equations

In many applications the experimentally measured trajectories of systems modeled by ODE do not in fact behave as predicted. Hence, it seems reasonable to modify ODE, somehow to include the possibility of random effects disturbingthe system. A formal way to do is by considering stochastic form of such DE's.

Definition 2.55. *[8]* ***Stochastic differential equations***
The Stochastic differential equations (SDE) defined

$$\begin{cases} \dot{\mathrm{X}}(t) = b(\mathrm{X}(t)) + B(\mathrm{X}(t))\xi(t), & t > 0 \\ \mathrm{X}(0) = x_0, \end{cases}$$

where B is (the space of $n \times m$ matrices)

$$B : \mathbb{R}^n \to \mathbb{M}^{n\times m},$$

and

$$\xi(\cdot) := m\text{-dimensional (white noise)}.$$

FIGURE 2.2: POSSIBLE SAMPLE PATH OF THE STOCHASTIC DIFFERENTIAL EQUATIONS.

Theorem 2.56. *[8]* ***Itô's Formula***
Let $X(\cdot)$ solves the stochastic differential

$$dX = Fdt + GdW.$$

Suppose $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous, where u_t, u_x and u_{xx} exists and continuous. Let Y solves the stochastic differential equation

$$Y(t) = u(X(t), t).$$

Then the Itô's formula or Itô's chain rule given by

$$\begin{aligned} dY &= u_t dt + u_x dX + \frac{1}{2} u_{xx} G^2 dt \\ &= \left(u_t + u_x F + \frac{1}{2} u_{xx} G^2 \right) dt + u_x G dW. \end{aligned}$$

That is, for all $0 \leq s \leq r \leq T$, we have

$$\begin{aligned} Y(r) - Y(s) &= u(X(r), r) - u(X(s), s) \\ &= \int_s^r u_t(X,t) + u_x(X,t)F + \frac{1}{2} u_{xx}(X,t) G^2 dt + \int_s^r u_x(X,t) G dW. \end{aligned}$$

Example 2.4. *Let $X(\cdot) = W(\cdot)$ with*

$$u(t) = e^{\lambda x - \frac{\lambda^2 t}{2}}.$$

Then $dX = dW$ and $F = 0, G = 1$.
Let

$$Y(t) = u(X(t), t) \Longrightarrow Y(0) = 1.$$

Thus, Itô's formula gives

$$\begin{aligned} dY =& \left(u_t + u_x F + \frac{1}{2} u_{xx} G^2 \right) dt + u_x G dW \\ =& \left(\frac{-\lambda^2}{2} e^{\lambda x - \frac{\lambda^2 t}{2}} + 0 + \frac{\lambda^2}{2} e^{\lambda x - \frac{\lambda^2 t}{2}} \right) dt + \lambda e^{\lambda x - \frac{\lambda^2 t}{2}} dW \\ =& \lambda e^{\lambda x - \frac{\lambda^2 t}{2}} dW \\ =& \lambda Y dW. \end{aligned}$$

Thus, the SDE is

$$\begin{cases} dY = \lambda Y dW, \\ Y(0) = 1, \end{cases}$$

has solution

$$Y(t) = e^{\lambda x - \frac{\lambda^2 t}{2}}.$$

2.3 Gompertz distribution

In this section we introduce the Gompertz distribution which is used to describe the distribution of human mortality.

Definition 2.57. *[26]* ***Gompertz distribution***
The Gompertz distribution has a continuous probability density function with location parameter $a \geq 0$ and shape parameter $b > 0$

$$f(x) = ae^{bx - \frac{a}{b}\left(e^{bx} - 1\right)}, \tag{2.1}$$

and $x \in (-\infty, \infty)$.

And the distribution function is

$$F(x) = 1 - e^{-\frac{a}{b}\left(e^{bx} - 1\right)}. \tag{2.2}$$

In actuarial or demographic applications, x usually denotes age which cannot be negative, leading to bounded support on $[0, \infty)$.

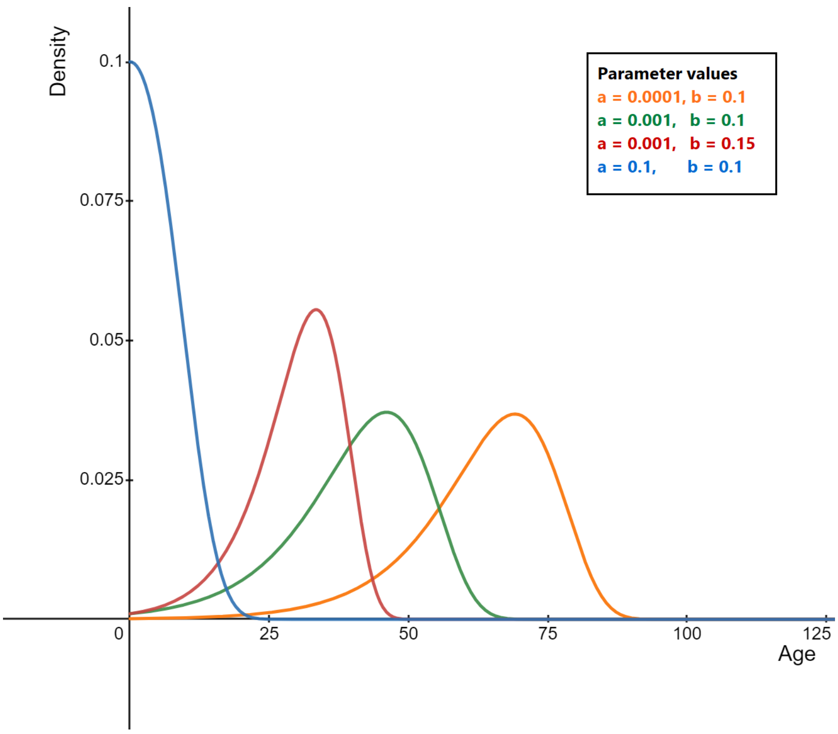

Figure 2.3: Gompertz density function for different combination of a and b parameters.

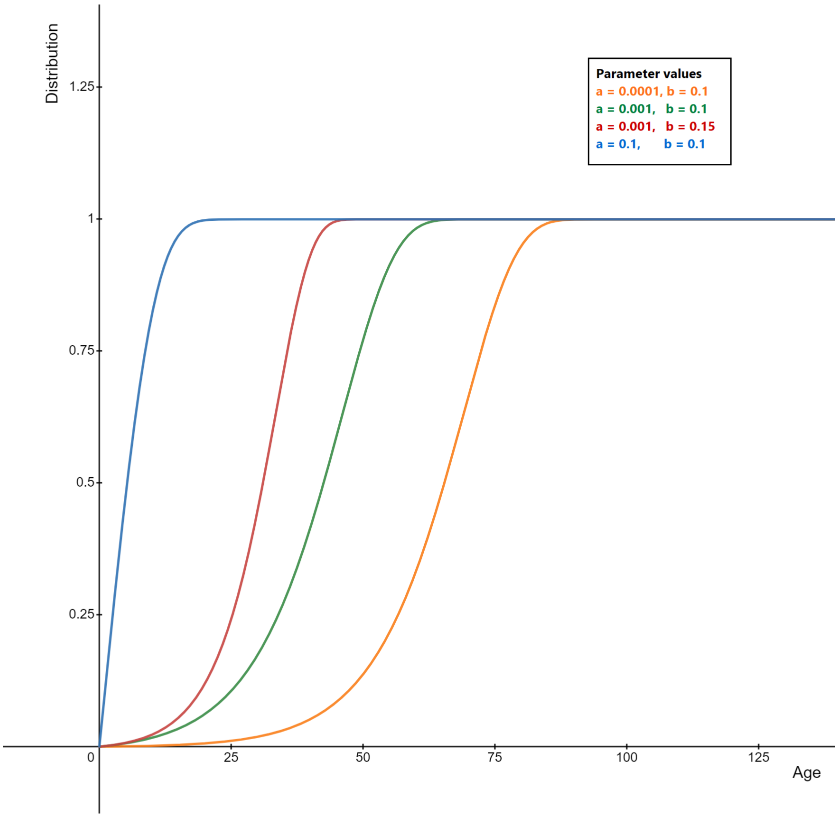

FIGURE 2.4: GOMPERTZ DISTRIBUTION FUNCTION FOR DIFFERENT COMBINATION OF a AND b PARAMETERS.

And the Gompertz force of mortality (or hazard function) $\lambda(x)$ *at age* $x \geq 0$ *is*

$$\lambda(x) = ae^{bx}, \tag{2.3}$$

where a *denotes the level of the force of mortality at age* 0 *and* b *the rate of aging.*

Remark 2.4. *[31] The Gompertz probability density function can be reformulated into a distribution with location and scale parameters. By substituted Gompertz parameters as*

$$b = \frac{1}{n} \quad \text{where } n > 0, \tag{2.4}$$

and

$$a = \frac{1}{n}e^{-\frac{m}{n}} \quad \text{where } m \in \mathbb{R}, \tag{2.5}$$

when substituted a *and* b *from equations* (2.4) *and* (2.5) *in equation* (2.1) *then the Gompertz density function can be expressed as*

$$f(x) = \frac{1}{n}e^{e^{-\frac{m}{n}} - e^{\frac{(x-m)}{n}} + \frac{(x-m)}{n}}. \tag{2.6}$$

Similarly for the distribution function from equation (2.2) we get

$$F(x) = 1 - e^{e^{-\frac{m}{n}} - e^{\frac{(x-m)}{n}}}. \tag{2.7}$$

And the hazard function (2.3) given as

$$\lambda(x) = \frac{1}{n} e^{\frac{(x-m)}{n}}. \tag{2.8}$$

2.4 Copula functions

In this section we will introduce a Copula functions and some special types. Origin of the word Copula is the latin word Copulare, which means (to join together).

Definition 2.58. *[36]* ***Uniform distribution***
A uniform distribution of X on $[a, b]$ is distribution with probability density function as

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b,$$

and its cumulative distribution function is

$$F(x) = \int_a^x f(t)dt = \frac{x-a}{b-a}, \quad a \le x \le b.$$

Definition 2.59. *[13]* ***n-dimensional Copula***
The n-dimensional Copula is a function $C : [0,1]^n \to [0,1]$ is a distribution function with uniform marginals. A Copula can be defined as a probability function

$$C(u_1, \ldots, u_n) = P(U_1 \le u_1, \ldots, U_n \le u_n),$$

where $U_1, \ldots, U_n$ are uniform distributions on the interval $[0, 1]$.

Proposition 2.1. *[13] Some properties for a Copula function*

1. *$C(u_1, \ldots, u_n)$ is increasing in each component u_i.*
2. *$C(1, \ldots, 1, u_m, 1, \ldots, 1) = u_m$ for every $m \in \{1, \ldots, n\}, u_m \in [0, 1]$.*
3. *$C(u_1, \ldots, u_n) = 0$ if $u_m = 0$ for every $m \le n$.*

Theorem 2.60. *[13]* ***Sklar's Theorem***
Assume F be a n-dimensional joint distribution function with marginals $F_1, \ldots, F_n$.

Then there exists a Copula $C : [0,1]^n \to [0,1]$ *such that*

$$F\left(x_1, \ldots, x_n\right) = C\left(F_1\left(x_1\right), \ldots, F_n\left(x_n\right)\right), \forall x_1, \ldots, x_n.$$

Remark 2.5. *[48] In 2 dimensional space, let* X *and* Y *be a pair of random variable with cumulative distribution function as* $F_X(x)$ *and* $F_Y(y)$*. Also let their joint cumulative distribution function is* $F_{X,Y}(x,y)$ *(this function is known as a two-dimensional Copula).*

Each pair (x,y) leads to a point in the unit square $[0,1] \times [0,1]$*. And this ordered pair in turn corresponds to a number* $F_{X,Y}(x,y)$ *in interval* $[0,1]$ *as in figure* (2.5).

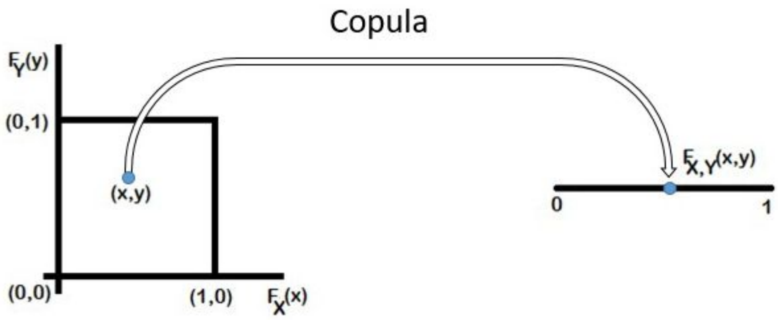

FIGURE 2.5: TWO-DIMENSIONAL COPULA.

Definition 2.61. *[48]* ***Copula density function***
The Copula density function c *is derived as*

$$c(s,t) = \frac{\partial^2 C(s,t)}{\partial s \partial t},$$

where $s = F_1(s)$ *and* $t = F_2(t)$ *are marginal functions.*

Definition 2.62. *[39]* ***Joint Copula density function***
If the bivariate joint distribution function $F(s,t)$ *is defined by a Copula function*

$$F(s,t) = C(F_1(s), F_2(t)),$$

where $s, t > 0$ *and* $F_1(s)$*,* $F_2(t)$ *are marginal functions.*
Then the joint density function of Copula is

$$f(s,t) = f_1(s) f_2(t) c(F_1(s), F_2(t)),$$

where c *is Copula density function.*

Example 2.5. ***Independence Copula***
Let the random variables X_1 and X_2 are independent.
The corresponding Copula is

$$C(u_1, u_2) = P(U_1 \leq u_1, U_2 \leq u_2) = P(U_1 \leq u_1)P(U_2 \leq u_2) = u_1 u_2.$$

The second equality is due to independence of U_1 and U_2, and the last equality is because U_1 and U_2 follow uniform distributions.

Example 2.6. ***Comonotonicity Copula***
Let $X_2 = 2X_1$; that is, X_1 and X_2 have a deterministic and positive relationship.
We can derive the relation between the CDEs

$$F_1(x) = \mathbb{P}(X_1 \leq x) = \mathbb{P}(2X_1 \leq 2x) = \mathbb{P}(X_2 \leq 2x) = F_2(2x),$$

which leads to the fact that U_1 is equal to U_2

$$U_1 = F_1(X_1) = F_2(2X_1) = F_2(X_2) = U_2.$$

The Copula is

$$\begin{aligned} C(u_1, u_2) &= \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2) \\ &= \mathbb{P}(U_1 \leq u_1, U_1 \leq u_2) \\ &= \mathbb{P}(U_1 \leq \min\{u_1, u_2\}) \\ &= \min\{u_1, u_2\}. \end{aligned}$$

The comonotonicity Copula has perfect positive dependence. Note that $X_2 = 2X_1$ can be replaced by $X_2 = T(X_1)$ for any strictly increasing transformation T.

Example 2.7. ***Counter monotonicity Copula***
Similar to the previous example, let $X_2 = -2X_1$ and consider the perfect negative dependence

$$F_1(x) = \mathbb{P}(X_1 \leq x) = \mathbb{P}(-2X_1 \geq -2x) = \mathbb{P}(X_2 \geq -2x) = 1 - F_2(-2x).$$

And

$$U_1 = F_1(X_1) = 1 - F_2(-2X_1) = 1 - F_2(X_2) = 1 - U_2.$$

The Copula is

$$\begin{aligned} C(u_1, u_2) &= \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2) \\ &= \mathbb{P}(U_1 \leq u_1, 1 - U_1 \leq u_2) \\ &= \mathbb{P}(1 - u_2 \leq U_1 \leq u_1) \\ &= max\{u_1 + u_2 - 1, 0\}. \end{aligned}$$

2.5 Archimedean Copula

There are many families of Copula function, in this section we will discuss the Archimedean Copula family with examples.

Definition 2.63. *[39]* ***Archimedean Copula***
The Copula in the form

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v)),$$

is called Archimedean Copula, with generating function $\varphi(x) : [0,1] \to [0,\infty)$, where φ is a real valued function satisfying the following conditions

1. $\varphi(1) = 0$.
2. $\lim_{x\to 0} \varphi(x) = \infty$.
3. $\frac{\partial \varphi}{\partial x} < 0$ *for all* $x \in (0,1)$.
4. $\frac{\partial^2 \varphi}{\partial x^2} > 0$ *for all* $x \in (0,1)$.

Example 2.8. *Let $\varphi(t) = 1 - t, t \in [0,1]$. Then the inverse is*

$$\varphi^{-1}(t) = \begin{cases} 1-t & ,0 \le t \le 1, \\ 0 & ,1 \le t < \infty. \end{cases}$$

Therefore, the Archimedean Copula is

$$\begin{aligned} C(u,v) &= \varphi^{-1}(\varphi(u) + \varphi(v)) \\ &= \varphi^{-1}(2-u-v) \\ &= \begin{cases} u+v-1 & ,0 \le 2-u-v \le 1 \\ 0 & ,1 \le 2-u-v, \end{cases} \\ &= \max(u+v-1, 0). \end{aligned}$$

In this Thesis, we will focus on the following three types of Archimedean Copulas.

Definition 2.64. *[12]* ***Frank Archimedean Copula***
Frank Archimedean Copula is

$$C(s,t) = \frac{1}{\alpha} \ln \left(1 + \frac{\left(e^{\alpha s} - 1\right)\left(e^{\alpha t} - 1\right)}{e^{\alpha} - 1} \right), \quad \alpha \neq 0,$$

where the generator function is $\varphi(t) = \ln \left(\frac{e^{\alpha t} - 1}{e^{\alpha} - 1} \right)$.

And, from the Frank Archimedean Copula Definition 2.64 we have

$$\frac{\partial C(s,t)}{\partial t} = \frac{(e^{\alpha s}-1)e^{\alpha t}}{(e^{\alpha}-1)+(e^{\alpha s}-1)(e^{\alpha t}-1)}. \tag{2.9}$$

Proposition 2.2. *[12]* ***Density function of Frank Copula***
Based on Definition 2.61 the density function of Frank Copula can be derived as

$$\begin{aligned} c(s,t) &= \frac{\partial^2 C(s,t)}{\partial s \partial t} \\ &= \frac{\partial}{\partial s}\left(\frac{(e^{\alpha s}-1)e^{\alpha t}}{(e^{\alpha}-1)+(e^{\alpha s}-1)(e^{\alpha t}-1)}\right) \\ &= \frac{\alpha(e^{\alpha}-1)e^{\alpha(t+s)}}{\left((e^{\alpha}-1)+(e^{\alpha s}-1)(e^{\alpha t}-1)\right)^2}. \end{aligned}$$

Definition 2.65. *[5]* ***Clayton Archimedean Copula***
Clayton Archimedean Copula is

$$C(s,t) = \left(s^{-\alpha}+t^{-\alpha}-1\right)^{\frac{-1}{\alpha}}, \alpha > 0,$$

where the generator function is $\varphi(t) = \frac{t^{-\alpha}-1}{\alpha}$.

And, from the Clayton Archimedean Copula Definition 2.65 we have

$$\frac{\partial C(s,t)}{\partial t} = t^{-(1+\alpha)}\left(s^{-\alpha}+t^{-\alpha}-1\right)^{-\frac{1+\alpha}{\alpha}}. \tag{2.10}$$

Proposition 2.3. *[5]* ***Density function of Clayton Copula***
The Copula density function of Clayton Copula can be derived as

$$c(s,t) = (1+\alpha)(st)^{-1-\alpha}\left(s^{-\alpha}+t^{-\alpha}-1\right)^{\frac{-1}{\alpha}-2}.$$

Definition 2.66. *[19]* ***Stable Archimedean Copula***
Stable (or Gumbel-Hougaard) Archimedean Copula is

$$C(s,t) = e^{-((-\ln s)^{\alpha}+(-\ln t)^{\alpha})^{\frac{1}{\alpha}}}, \alpha \geq 1$$

, where the generator function is $\varphi(t) = (-\ln t)^{\alpha}$.

And, from the Gumbel-Hougaard Archimedean Copula Definition 2.66 we have

$$\frac{\partial C(s,t)}{\partial t} = \frac{(-\ln t)^{\alpha-1}}{t} e^{-((-\ln s)^{\alpha}+(-\ln t)^{\alpha})^{\frac{1}{\alpha}}}\left((-\ln s)^{\alpha}+(-\ln t)^{\alpha}\right)^{\frac{1-\alpha}{\alpha}}. \tag{2.11}$$

Proposition 2.4. *[19] **Density function of Stable Copula***
The Copula density function of Stable Copula can be derived as

$$
\begin{aligned}
c(s,t) = {} & e^{-((-\ln s)^{\alpha}+(-\ln t)^{\alpha})^{\frac{1}{\alpha}}}((-\ln s)^{\alpha}+(-\ln t)^{\alpha})^{-2+\frac{2}{\alpha}} \\
& \times \left(1+(\alpha-1)\left((-\ln s)^{\alpha}+(-\ln t)^{\alpha}\right)^{\frac{1}{\alpha}}\right)(st)^{-1}\left(\ln s . \ln t\right)^{\alpha-1}.
\end{aligned}
$$

2.6 Archimedean Copula on Gompertz distribution

In this section, we will give an example, by applying a Copulas functions on Gompertz distribution defined in equations (2.6) and (2.7). Three types of Archimedean Copulas are introduced.

To proceed we find the following integrals

$$\int_{t}^{\infty} f(s,t)\mathrm{d}s, \tag{2.12}$$

and

$$\int_{t}^{\infty} f(t,z)\mathrm{d}z, \tag{2.13}$$

as in the following lemmas.

In the next lemma we give the formula of the integral (2.12) that we are looking for, using three different types of Archimedean Copulas.

Lemma 2.67. *Assume the bivariate joint distribution function $F(s,t)$ is defined by a Copula function*

$$F(s,t) = C(F_1(s), F_2(t)), \tag{2.14}$$

where $s,t > 0$ and each marginal distribution is represent to a Gompertz distribution. Then the values of the integral (2.12) are given by

(i) ***Frank Copula***

$$\frac{f_2(t)(e^{\alpha}-e^{\alpha F_1(t)})}{(e^{\alpha}-1)+\left(e^{\alpha F_1(t)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}.$$

(ii) ***Clayton Copula***

$$f_2(t) - f_2(t)\left(1+\left(\frac{F_2(t)}{F_1(t)}\right)^{\alpha}-(F_2(t))^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}}.$$

(iii) ***Stable Copula***

$$f_2(t) - \frac{f_2(t)F(t,t)}{F_2(t)}\left(\left(\frac{\ln F_1(t)}{\ln F_2(t)}\right)^{\alpha} + 1\right)^{\frac{1-\alpha}{\alpha}}.$$

Proof. From equations (2.6) and (2.7), for $i = 1, 2$, we have

$$F_i(t) = 1 - e^{e^{-\frac{m_i}{n_i}} - e^{\frac{(t-m_i)}{n_i}}},$$

and

$$f_i(t) = \frac{1}{n_i} e^{e^{-\frac{m_i}{n_i}} - e^{\frac{(t-m_i)}{n_i}} + \frac{(t-m_i)}{n_i}}, \tag{2.15}$$

with m_i and n_i, are parameters. The hazard function from equation (2.8) is given as

$$\lambda_i(t) = \frac{1}{n_i} e^{\frac{(t-m_i)}{n_i}}.$$

Since $F(s,t)$ is bivariate joint distribution function, it follows that from Definition 2.62, the corresponding joint density function $f(s,t)$ given by

$$f(s,t) = f_1(s)f_2(t)c(F_1(s), F_2(t)),$$

where c is the Copula density function. Substituting the value of $f(s,t)$ from (2.67) in the integral (2.12), we get

$$\int_t^{\infty} f_1(s)f_2(t)c(F_1(s), F_2(t))\mathrm{d}s, \tag{2.16}$$

where that, $f_1(s)$ is the density function of $F_1(s)$ given in equation (2.15). Hence,

$$\frac{\mathrm{d}F_1(s)}{\mathrm{d}s} = f_1(s),$$

and

$$\mathrm{d}F_1(s) = f_1(s)\mathrm{d}s. \tag{2.17}$$

Substitute equation (2.17) in the integral (2.16) to obtain

$$\int_t^{\infty} f_2(t)c(F_1(s), F_2(t))\mathrm{d}F_1(s).$$

Substitute the value of Copula density function from Definition 2.61 in above integral to obtain

$$f_2(t)\int_t^{\infty} \frac{\partial^2 C((F_1(s), F_2(t)))}{\partial(F_1(s))\partial(F_2(t))}\mathrm{d}F_1(s). \tag{2.18}$$

Integrate (2.18) to get

$$f_2(t)\left(\frac{\partial C((F_1(s),F_2(t)))}{\partial(F_2(t))}\Bigg|_{s=t}^{s=\infty}\right). \tag{2.19}$$

Now to find (2.19) we will apply three types of Archimedean copulas.

(i) **Frank Copula**
Substitute the value of $\frac{\partial C((F_1(s),F_2(t)))}{\partial(F_2(t))}$ from equation (2.9) in (2.19), we get

$$f_2(t)\left(\frac{(e^{\alpha F_1(s)}-1)e^{\alpha F_2(t)}}{(e^{\alpha}-1)+\left(e^{\alpha F_1(s)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}\Bigg|_{s=t}^{s=\infty}\right).$$

Simplify,

$$f_2(t)\lim_{b\to\infty}\left(\frac{(e^{\alpha F_1(b)}-1)e^{\alpha F_2(t)}}{(e^{\alpha}-1)+\left(e^{\alpha F_1(b)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}\right.$$
$$\left.-\frac{(e^{\alpha F_1(t)}-1)e^{\alpha F_2(t)}}{(e^{\alpha}-1)+\left(e^{\alpha F_1(t)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}\right).$$

The Gompertz distribution function F_1 goes to 1 as $b\to\infty$, so the above terms becomes

$$f_2(t)\left(\frac{(e^{\alpha}-1)e^{\alpha F_2(t)}}{(e^{\alpha}-1)+(e^{\alpha}-1)\left(e^{\alpha F_2(t)}-1\right)}-\frac{(e^{\alpha F_1(t)}-1)e^{\alpha F_2(t)}}{(e^{\alpha}-1)+\left(e^{\alpha F_1(t)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}\right).$$

Rearrange the above terms we get

$$f_2(t)\left(\frac{(e^{\alpha}-1)e^{\alpha F_2(t)}}{(e^{\alpha}-1)\left(1+e^{\alpha F_2(t)}-1\right)}-\frac{(e^{\alpha F_1(t)}-1)e^{\alpha F_2(t)}}{(e^{\alpha}-1)+\left(e^{\alpha F_1(t)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}\right).$$

That is,

$$f_2(t)\left(1-\frac{(e^{\alpha F_1(t)}-1)e^{\alpha F_2(t)}}{(e^{\alpha}-1)+\left(e^{\alpha F_1(t)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}\right). \tag{2.20}$$

Rearrange (2.20) we get

$$f_2(t)\left(\frac{e^{\alpha}-1+e^{\alpha F_1(t)}e^{\alpha F_2(t)}-e^{\alpha F_1(t)}-e^{\alpha F_2(t)}+1-e^{\alpha F_1(t)}e^{\alpha F_2(t)}+e^{\alpha F_2(t)}}{(e^{\alpha}-1)+\left(e^{\alpha F_1(t)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}\right).$$

Hence, by the Frank Copula, the integral (2.12) is equal to

$$\frac{f_2(t)(e^{\alpha}-e^{\alpha F_1(t)})}{(e^{\alpha}-1)+\left(e^{\alpha F_1(t)}-1\right)\left(e^{\alpha F_2(t)}-1\right)}.$$

(ii) **Clayton Copula**

From equation (2.10), the first derivative of Clayton Copula can be written as

$$\frac{\partial C(s,t)}{\partial t} = \left(1 + \left(\frac{t}{s}\right)^{\alpha} - t^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}}. \tag{2.21}$$

Substitute $F_1(s)$ and $F_2(t)$ in equation (2.21) to get

$$\frac{\partial C(F_1(s),F_2(t))}{\partial(F_2(t))} = \left(1 + \left(\frac{F_2(t)}{F_1(s)}\right)^{\alpha} - (F_2(t))^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}}. \tag{2.22}$$

Substitute equation (2.22) in equation (2.19) we get

$$f_2(t)\left(\left(1 + \left(\frac{F_2(t)}{F_1(s)}\right)^{\alpha} - (F_2(t))^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}}\Bigg|_{s=t}^{s=\infty}\right).$$

Thus,

$$f_2(t)\lim_{b\to\infty}\left(\left(1 + \left(\frac{F_2(t)}{F_1(b)}\right)^{\alpha} - (F_2(t))^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}} - \left(1 + \left(\frac{F_2(t)}{F_1(t)}\right)^{\alpha} - (F_2(t))^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}}\right).$$

The Gompertz distribution function F_1 goes to 1 as $b \to \infty$, so the above terms becomes

$$f_2(t)\left((1 + (F_2(t))^{\alpha} - (F_2(t))^{\alpha})^{-\frac{1+\alpha}{\alpha}} - \left(1 + \left(\frac{F_2(t)}{F_1(t)}\right)^{\alpha} - (F_2(t))^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}}\right).$$

Hence, by the Clayton Copula, the integral (2.12) is equal to

$$f_2(t) - f_2(t)\left(1 + \left(\frac{F_2(t)}{F_1(t)}\right)^{\alpha} - (F_2(t))^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}}.$$

(iii) **Stable Copula**

From equation (2.11), the first derivative of Stable Copula can be written as

$$\frac{\partial C(s,t)}{\partial t} = \frac{1}{t}e^{-((-\ln s)^{\alpha}+(-\ln t)^{\alpha})^{\frac{1}{\alpha}}}\left(\left(\frac{\ln s}{\ln t}\right)^{\alpha} + 1\right)^{\frac{1-\alpha}{\alpha}}. \tag{2.23}$$

Substitute $F_1(s)$ and $F_2(t)$ in equation (2.23) to get

$$\frac{\partial C(F_1(s),F_2(t))}{\partial(F_2(t))} = \frac{1}{F_2(t)}e^{-((-\ln F_1(s))^{\alpha}+(-\ln F_2(t))^{\alpha})^{\frac{1}{\alpha}}}\left(\left(\frac{\ln F_1(s)}{\ln F_2(t)}\right)^{\alpha} + 1\right)^{\frac{1-\alpha}{\alpha}}. \tag{2.24}$$

Substitute equation (2.24) in equation (2.19), we get

$$f_2(t)\left(\frac{1}{F_2(t)}e^{-((-\ln F_1(s))^\alpha+(-\ln F_2(t))^\alpha)^{\frac{1}{\alpha}}}\left(\left(\frac{\ln F_1(s)}{\ln F_2(t)}\right)^\alpha+1\right)^{\frac{1-\alpha}{\alpha}}\Bigg|_{s=t}^{s=\infty}\right).$$

Thus,

$$f_2(t)\lim_{b\to\infty}\left(\frac{1}{F_2(t)}e^{-((-\ln F_1(b))^\alpha+(-\ln F_2(t))^\alpha)^{\frac{1}{\alpha}}}\left(\left(\frac{\ln F_1(b)}{\ln F_2(t)}\right)^\alpha+1\right)^{\frac{1-\alpha}{\alpha}}\right)$$
$$-f_2(t)\left(\frac{1}{F_2(t)}e^{-((-\ln F_1(t))^\alpha+(-\ln F_2(t))^\alpha)^{\frac{1}{\alpha}}}\left(\left(\frac{\ln F_1(t)}{\ln F_2(t)}\right)^\alpha+1\right)^{\frac{1-\alpha}{\alpha}}\right).$$

The Gompertz distribution function F_1 goes to 1 as $b\to\infty$, so the above terms becomes

$$f_2(t)-f_2(t)\left(\frac{1}{F_2(t)}e^{-((-\ln F_1(t))^\alpha+(-\ln F_2(t))^\alpha)^{\frac{1}{\alpha}}}\left(\left(\frac{\ln F_1(t)}{\ln F_2(t)}\right)^\alpha+1\right)^{\frac{1-\alpha}{\alpha}}\right). \tag{2.25}$$

From Definition of Stable Copula 2.66, we know that

$$C(F_1(t),F_2(t))=e^{-((-\ln F_1(t))^\alpha+(-\ln F_2(t))^\alpha)^{\frac{1}{\alpha}}}. \tag{2.26}$$

Substitute equation (2.26) in (2.25) to get

$$f_2(t)-\frac{f_2(t)C(F_1(t),F_2(t))}{F_2(t)}\left(\left(\frac{\ln F_1(t)}{\ln F_2(t)}\right)^\alpha+1\right)^{\frac{1-\alpha}{\alpha}}. \tag{2.27}$$

Substitute equation (2.14) in (2.27)

$$f_2(t)-\frac{f_2(t)F(t,t)}{F_2(t)}\left(\left(\frac{\ln F_1(t)}{\ln F_2(t)}\right)^\alpha+1\right)^{\frac{1-\alpha}{\alpha}},$$

which is a result of the integral in (2.12) by the Stable Copula.

This completes the proof. □

Next lemma gives us the formula of the integral (2.13) we are looking for using three different types of Archimedean Copulas

Lemma 2.68. *Assume the bivariate joint distribution function $F(s,t)$ is defined by a Copula function*

$$F(s,t)=C(F_1(s),F_2(t)),$$

where $s, t > 0$ and each marginal distribution is represent to a Gompertz distribution. Then the values of the integral (2.13) are given by

(i) ***Frank Copula***

$$\int_t^{\infty} f(t,z)\mathrm{d}z = \frac{f_2(t)(e^{\alpha} - e^{\alpha F_1(t)})}{(e^{\alpha}-1) + \left(e^{\alpha F_1(t)} - 1\right)\left(e^{\alpha F_2(t)} - 1\right)}.$$

(ii) ***Clayton Copula***

$$\int_t^{\infty} f(t,z)\mathrm{d}z = f_2(t) - f_2(t)\left(1 + \left(\frac{F_2(t)}{F_1(t)}\right)^{\alpha} - (F_2(t))^{\alpha}\right)^{-\frac{1+\alpha}{\alpha}}.$$

(iii) ***Stable Copula***

$$\int_t^{\infty} f(t,z)\mathrm{d}z = f_2(t) - \frac{f_2(t)F(t,t)}{F_2(t)}\left(\left(\frac{\ln F_1(t)}{\ln F_2(t)}\right)^{\alpha} + 1\right)^{\frac{1-\alpha}{\alpha}}.$$

Proof. Similar to the proof of Lemma 2.67. □

Chapter 3

Optimal welfare strategies for a two-wage earners within a markets of life-insurance and welfare providers

In this chapter, we extend the work done by Wei et al introduced in [45] by adding the welfare policy to be an additional control variable on the problem for the two wage-earners. We suppose that the two wage-earners are entered the welfare markets to protect there families through a social welfare provider which is available to both agents.

3.1 Model setup

In this section, our industrial markets consists of the financial market which is available for the two wage-earners, the life insurance market and social welfare market. We describe their details separately. After that, we introduce the corresponding wealth process.

3.1.1 Financial market model

We will introduce the financial market and make it available for investment all times and consists as follows

- $T < \infty$ denotes the common retirement time of the two wage-earners.
- τ_i represent to death time of wage-earner i where $i = 1, 2$, and (τ_1, τ_2), that are independent of each other.
- $W(\cdot)$ represent to standard Brownian motion.
- $T_1 = \tau_1 \wedge \tau_2$ represent the time of death of the first wage-earner from the two wage-earners.
- A probability space
$$(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P}),$$
where the natural filtration
$$\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]},$$
is generated by $W(\cdot)$.
- The first wage-earner earns a deterministic labor incomes of $I_1(\cdot)$.
- The second wage-earner earns a deterministic labor incomes of $I_2(\cdot)$.
- The risk-free return rate $r(\cdot)$ is such that $r(t) > 0$.
- The appreciation rate of the risky asset $\mu(\cdot)$ is defined by
$$\mu : [0, T] \to \mathbb{R}.$$
- The volatility of the risky asset $\sigma(\cdot)$ is defined by
$$\sigma : [0, T] \to \mathbb{R}.$$
- The financial market has two types of assets, riskless $B(\cdot)$ and risky $S(\cdot)$ with prices satisfying the following differential equations, respectively,
$$\begin{aligned} \frac{\mathrm{d}B(t)}{B(t)} &= r(t)\mathrm{d}t, \\ \frac{\mathrm{d}S(t)}{S(t)} &= \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \end{aligned} \tag{3.1}$$
where t from 0 up until the retirement time T.

Assumption 1. *1. The coefficients $r(t)$, $\mu(t)$ and $\sigma(t)$ are assumed to be deterministic continuous functions on interval $[0, T]$.*

2. *There exists an $\{\mathcal{F}_t\}_{0\leq t\leq T}$ progressively measurable process $\phi(t) \in \mathbb{R}$, called the market price of risk, such that for Lebesgue-almost-every $t \in [0, T]$, the risk premium*

$$\beta(t) = (\mu\,(t) - r\,(t)) \in \mathbb{R},$$

is connected to the market price of risk $\phi(t)$ by the identity

$$\beta(t) = \sigma(t)\phi(t), \quad a.s.$$

3. *$\sigma(t)$ must satisfies the following condition*

$$\int_0^T \sigma^2\,(t)\,dt < \infty.$$

4. *We assume that*

$$\int_0^T \phi^2(t)\mathrm{d}t < \infty \quad a.s.$$

5. *The following exponential martingale condition holds*

$$\mathbb{E}_t[e^{-\int_0^T \phi(t)\mathrm{d}W(t) - \frac{1}{2}\int_0^T \|\phi(t)\|^2\mathrm{d}t}] = 1.$$

6. *Assume that the lifetime of the two wage-earners are being random and represented by a continuous random variable ($\tau_1 > 0$, $\tau_2 > 0$) defined on the probability space*

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P}).$$

3.1.2 Life-insurance market model

In this section, we will define the distribution of the random variables (τ_1, τ_2), and the corresponding conditional probability distribution functions to help us introducing the life-insurance market model. To do so, we assume that the two wage-earners have a participated in the life-insurance market by paying an amount of premium $k_i(t)$ to the life-insurance provider, where $t \in [0, \tau_i \wedge T]$.

Assumption 2. *The random variable τ_i for the wage-earner i has a distribution function $F_i : [0, \infty) \to [0, 1]$ is given by*

$$F_i(t) = P(\tau_i \leq t) = \int_0^t f_i(s)\,\mathrm{d}s,$$

where f_i is the density function defined by

$$f_i : [0, \infty) \to \mathbb{R}^+.$$

Now we define a new function, which is called the survival function

$$F_i^+ : [0, \infty) \to [0, 1],$$

for the wage-earner i to survive past time t is defined as

$$F_i^+(t) = P(\tau_i > t) = 1 - F_i(t), \quad i = 1, 2.$$

Assume that the mortality rate (or hazard function) $\lambda_i(t)$ for the wage-earner i defined as

$$\lambda_i(t) = \lim_{\Delta t \to 0} \frac{P(t < \tau_i \leq t + \Delta t | \tau_i > t)}{\Delta t}, \quad i = 1, 2, \tag{3.2}$$

which represents the conditional instantaneous death rate for the wage-earner surviving past time t.

From equation (3.2) and Assumption 2 we have

$$\begin{aligned}
\lambda_i(t) &= \lim_{\Delta t \to 0} \frac{P(t < \tau_i \leq t + \Delta t | \tau_i > t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{P(\tau_i \leq t + \Delta t) - P(\tau_i \leq t)}{\Delta t P(\tau_i > t)} \\
&= \lim_{\Delta t \to 0} \frac{F_i(t + \Delta t) - F_i(t)}{\Delta t P(\tau_i > t)} \\
&= \frac{1}{P(\tau_i > t)} \lim_{\Delta t \to 0} \frac{F_i(t + \Delta t) - F_i(t)}{\Delta t} \\
&= \frac{f_i(t)}{F_i^+(t)} \\
&= -\frac{d}{dt}(\ln F_i^+(t)).
\end{aligned}$$

Hence, we can write $F_i^+(t)$ in the form

$$P(\tau_i > t) = F_i^+(t) = e^{-\int_0^t \lambda_i(z) \mathrm{d}z}.$$

Thus, the probability distribution functions $F_i(t)$ is

$$\mathrm{P}\left(\tau_i \leq t\right) = F_i(t) = 1 - e^{-\int_0^t \lambda_i(z) \mathrm{d}z}. \tag{3.3}$$

And $f_i(t)$ is related to $\lambda_i(t)$ by the identity

$$f_i(t) = \lambda_i(t) F_i^+(t) = \lambda_i(t) e^{-\int_0^t \lambda_i(z) \mathrm{d}z}. \tag{3.4}$$

Similarly, we can define conditional probability distribution functions as follows

- The conditional probability function $F_1(s;t)$ of wage-earner 1 die before time s, given that he is alive at time t and $t \leq s$ is

$$F_1(s;t) = P(\tau_1 \leq s | \tau_1 > t). \tag{3.5}$$

  The corresponding conditional density function of $F_1(s;t)$ is $f_1(s;t)$.

- The conditional probability function $F_2(s;t)$ of wage-earner 2 die before time s, given that he is alive at time t and $t \leq s$ is

$$F_2(s;t) = P(\tau_2 \leq s | \tau_2 > t). \tag{3.6}$$

  The corresponding conditional density function of $F_2(s;t)$ is $f_2(s;t)$.

- The conditional probability function $F_{T_1}(s;t)$ represent to first death between the two wage-earners occur before time s, given that they are alive at time t and $t \leq s$ is

$$F_{T_1}(s;t) = P(T_1 \leq s | T_1 > t). \tag{3.7}$$

  The corresponding conditional density function of $F_{T_1}(s;t)$ is $f_{T_1}(s;t)$.

- The conditional probability function $F(s_1, s_2; t)$ represent to the wage-earner i dies before time s_i where i = 1 and 2, given that they are alive at time t and $t \leq \min\{s_1, s_2\}$ is

$$F(s_1, s_2; t) = P(\tau_1 \leq s_1, \tau_2 \leq s_2 | T_1 > t), \tag{3.8}$$

  The corresponding conditional density function of $F(s_1, s_2; t)$ is $f(s_1, s_2; t)$.

For $i = 1, 2$, any contract between the wage-earner i and the insurance company starts at time $t = 0$ and stops at time equals to the minimum between the death time of the wage-earner τ_i or it's retirement time T; i.e., $(\tau_i \wedge T)$. If the wage-earner i dies at time τ_i while having a contract with the insurance company by buying life-insurance premium rates $k_i(\tau_i)$, then that insurance company pays an amount

$$\frac{k_i(\tau_i)}{\eta_i(\tau_i)},$$

where

$$\eta_i : [0, T] \to \mathbb{R}^+$$

denotes the premium-insurance ratio which is determined by the insurance company.
In case $\tau_i < \tau_{3-i}(i = 1, 2)$, which means the wage-earner i dies first, the total wealth at death time τ_i jumps to

$$X(\tau_i) = X(\tau_i -) + \frac{k_i(\tau_i)}{\eta_i(\tau_i)},$$

where $X(\tau_i -)$ denotes the wealth for the wage-earner i prior the time of death.
If the two wage-earners dies simultaneously, then the insurance company pays to there beneficiary an amount

$$\sum_{i=1}^{2} \frac{k_i(\tau_i)}{\eta_i(\tau_i)}.$$

3.1.3 Social security system model

In this section, we will allow the two wage-earners to contribute in the social security system in order to protect their families in the future. We assume that the wage-earner $i, i = 1, 2$ contributes in the welfare system by paying an amount $q_i(t)$ to the welfare provider, where the time t is such that $t \in [0, \tau_i \wedge T]$.

If the wage-earner i dies at time $\tau_i \leq T$, while he is participating in the social security system, then the social security company has to pay to his beneficiary the amount

$$\frac{q_i(t)}{h_i(t)},$$

where

$$h_i(t) : [0, T] \to \mathbb{R}^+,$$

is a continuous and deterministic positive function which is determined by the social security company, and

$$q_i(t) : [0, T] \to \mathbb{R}^+,$$

is a non-negative deterministic function.

The participation in the social system ends when the wage-earner dies or achieves retirement age, whichever occurs first. Therefore, for the wage-earner i, the total wealth at death time $\tau_i \leq T$ equals to

$$\bar{X}(\tau_i) = X(\tau_i -) + \frac{k_i(\tau_i)}{\eta_i(\tau_i)} + \frac{q_i(\tau_i)}{h_i(\tau_i)}.$$

3.2 Optimal control problem

In this section, we will consider the optimal control problem for the two wage-earners whose goal is find the optimal strategies that will maximize the expected utility, while the two wage-earners are purchasing life-insurance and having access to the social security system.

To proceed in modeling the flow of the wage-earner's wealth, we first introduce the following assumption.

Assumption 3. *Let $t \in [0, min\{\tau, T\}]$. We assume the following*

- *The income function $I_i : [0,T] \to \mathbb{R}_0^+$, $i = 1,2$, is a deterministic Borel-measurable function satisfying the integrability condition*

$$\int_0^T I_i(t)\ \mathrm{d}t < \infty.$$

- *The consumption process $(c_i(t))_{0\le t\le T}$, $i = 1,2$, is a $\{\mathcal{F}_t\}_{0\le t\le T}$ progressively measurable non-negative process satisfying the following integrability condition for the investment horizon $T > 0$*

$$\int_0^T c_i(t)\ \mathrm{d}t < \infty \qquad a.s..$$

- *The premium insurance rate $(k_i(t))_{0\le t\le T}$ and welfare premium payout $(q_i(t))_{0\le t\le T}$ are non-negative $\{\mathcal{F}_t\}_{0\le t\le T}$ predictable process, for $i = 1,2$.*

Let $u_i(\cdot)$ be the amount invested in the risky asset where $i = 1,2$. For the wage-earner i, the wealth process is defined as

$$\begin{aligned} X_i(t) &= x_{i,0} + \int_0^t \Big(-c_i(s) - k_i(s) - q_i(s) + I_i(s) \Big)\mathrm{d}s + \int_0^t \frac{(X_i(s) - u_i(s))}{B(s)}\mathrm{d}B(s) \\ &+ \int_0^t \frac{u_i(s)}{S(s)}\mathrm{d}S(s), \end{aligned}$$

where $x_{i,0}$ is the initial wealth of wage-earner i. The last equation can be written using the dynamics in equation (3.1) as

$$\begin{aligned} X_i(t) &= x_{i,0} + \int_0^t \Big(-c_i(s) - k_i(s) - q_i(s) + I_i(s) \Big)\mathrm{d}s + \int_0^t \Big(X_i(s) - u_i(s) \Big) r(s)\mathrm{d}s \\ &+ \int_0^t u_i(s)\Big(\mu(s)\mathrm{d}t + \sigma(s)\mathrm{d}W(s) \Big). \end{aligned}$$

After rearranging the above terms we can get the following form

$$\begin{aligned} X_i(t) &= x_{i,0} + \int_0^t \Big(r(s)X_i(s) + (\mu(s) - r(s))u_i(s) - c_i(s) - k_i(s) - q_i(s) + I_i(s) \Big) \mathrm{d}s \\ &+ \int_0^t u_i(s)\sigma(s)\mathrm{d}W(s). \end{aligned} \tag{3.9}$$

Differentiate equation (3.9) with respect to t to get this differential form

$$\begin{aligned} \mathrm{d}X_i(t) = &\left[r(t)X_i(t) + (\mu(t) - r(t))u_i(t) - c_i(t) - k_i(t) - q_i(t) + I_i(t) \right] \mathrm{d}t \\ &+ \sigma(t)u_i(t)\mathrm{d}W(t), \quad t \in [0, \tau_i \wedge T]. \end{aligned} \tag{3.10}$$

The total household wealth defined as

$$X(\cdot) = X_1(\cdot) + X_2(\cdot). \tag{3.11}$$

Thus, differentiate both sides

$$\mathrm{d}X(\cdot) = \mathrm{d}X_1(\cdot) + \mathrm{d}X_2(\cdot). \tag{3.12}$$

Define the total investment in risky asset as

$$u(t) = \sum_{i=1}^{2} u_i(t)\mathbf{1}_{\{t<\tau_i\}}, \tag{3.13}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

Substituting the identities of (3.10), (3.11) and (3.13) in the equation (3.12) we obtain

$$\begin{aligned} \mathrm{d}X(t) = &\Big[r(t)X(t) + [\mu(t) - r(t)]u(t) - \mathbf{1}_{\{t<\tau_1\}} \left[c_1(t) + k_1(t) + q_1(t) - I_1(t) \right] \\ &- \mathbf{1}_{\{t<\tau_2\}} \left[c_2(t) + k_2(t) + q_2(t) - I_2(t) \right] \Big] \mathrm{d}t + \sigma(t)u(t)\mathrm{d}W(t), \quad t \in [0, (\tau_1 \vee \tau_2) \wedge T]. \end{aligned}$$

3.3 Stochastic optimal control problem

In this section, we will state the expected utility and the value function of the two wage-earners using the dynamic of the wealth process. To proceed, assume $\mathcal{A}$ be the set of admissible strategies which has the form

$$\pi(\cdot) = (c_1(\cdot), c_2(\cdot), k_1(\cdot), k_2(\cdot), q_1(\cdot), q_2(\cdot), u(\cdot)),$$

where c_i is the consumption of agent i, k_i is the life-insurance premium rate of agent i, q_i is the welfare system premium rate of agent i, $i = 1, 2$, and u is the total investment in the risky asset.

The set of all admissible strategies $\mathcal{A}$ can be expressed as

$$\begin{aligned}\mathcal{A} = \Big\{ & \pi(\cdot) \in \left(\mathbb{R}^+\right)^7 \mid \pi(\cdot) \text{ is } \mathbb{F}- \text{ measurable },\\ & \mathbb{E}_t\left[\int_0^T u^2(t)\mathrm{d}t\right] < \infty,\\ & \mathbb{E}_t\left[\int_0^T |c_i(t)|\,\mathrm{d}t\right] < \infty,\\ & \mathbb{E}_t\left[\int_0^T |k_i(t)|\,\mathrm{d}t\right] < \infty,\\ & \mathbb{E}_t\left[\int_0^T |q_i(t)|\,\mathrm{d}t\right] < \infty, i = 1, 2,\end{aligned}$$

where $\mathbb{E}_t[\cdot]$ denote the expectation conditioned on $\mathcal{F}_t$.

For any $\pi \in \mathcal{A}$ we define

$$\begin{aligned}J(t, x; \pi(\cdot)) = \mathbb{E}_t\Bigg[& \int_t^{\tau_1 \wedge T} w_1 e^{-\delta s} U\left(c_1(s)\right) \mathrm{d}s\\ & + \int_t^{\tau_2 \wedge T} w_2 e^{-\delta s} U\left(c_2(s)\right) \mathrm{d}s + w_3 \mathbf{1}_{\{\tau_1 \vee \tau_2 \leq T\}} e^{-\delta(\tau_1 \vee \tau_2)}\\ & \times \left(U\left(X\left(\tau_1 \vee \tau_2\right) + \sum_{i=1}^{2} \frac{k_i\left(\tau_i\right)}{\eta_i\left(\tau_i\right)} \mathbf{1}_{\{\tau_i = \tau_1 \vee \tau_2\}} \right) + U\left(\sum_{i=1}^{2} \frac{q_i\left(\tau_i\right)}{h_i\left(\tau_i\right)} \mathbf{1}_{\{\tau_i = \tau_1 \vee \tau_2\}} \right) \right)\\ & + w_4 \mathbf{1}_{\{\tau_1 \vee \tau_2 > T\}} e^{-\delta T} U(X(T)) \Bigg],\end{aligned} \tag{3.14}$$

where $U(\cdot)$ represent the utility function and the constants

$$w_i \geq 0, \text{for all } i = 1, 2, 3, 4,$$

have sum to one unit. That is,

$$\sum_{i=1}^{4} w_i = 1.$$

The constants w_i are reflecting the relative importance of one utility type with respect to another utility and δ is discount factor.

Finally, the value function will be defined as

$$V(t, x) = \max_{\pi \in \mathcal{A}} J(t, x; \pi(\cdot)).$$

Chapter 4

Optimization problem after the first death

Based on Chapter 3, in this chapter we will find an explicit solution for the problem under consideration when a post death of one person from the two wage-earners has accrued. To proceed, we will first transform the stochastic optimal control problem of the two wage-earners to an equivalent one with fixed planning horizon, after that we can derive a dynamic programming principle (DPP) and the corresponding the HJB equation.

4.1 Stochastic optimal control problem after the first death

In this section, we will state the stochastic optimal control problem post first death, and make certain assumptions that help us to state our DPP.

To solve the optimization problem after the death of one wage-earner from the two wage-earners, we consider the following case

$$T_1 = \tau_{3-i} < \tau_i, \quad i = 1, 2.$$

If $T_1 = \tau_{3-i}$, after death of one person, the optimization problem of the wage-earner i can be considered as

$$
\begin{aligned}
\max \tilde{J}_i\left(t, x; c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot)\right) = \mathbb{E}_t \Bigg[\int_t^{\tau_i \wedge T} & w_i e^{-\delta s} U\left(c_i(s)\right) \mathrm{d}s \\
+ w_3 \mathbf{1}_{\{\tau_i \leq T\}} e^{-\delta \tau_i} & \left(U\left(X\left(\tau_i\right) + \frac{k_i\left(\tau_i\right)}{\eta_i\left(\tau_i\right)} \right) + U\left(\frac{q_i\left(\tau_i\right)}{h_i\left(\tau_i\right)} \right) \right) \\
+ w_4 \mathbf{1}_{\{\tau_i > T\}} e^{-\delta T} & U(X(T)) \Bigg],
\end{aligned}
\tag{4.1}
$$

where $X_i(\cdot)$ is as given in equation (3.9).
For $i = 1, 2$, the value functions $V_i(t, x)$ are expressed as

$$
V_i(t, x) = \sup_{(c_i, k_i, q_i, u_i)} \tilde{J}_i\left(t, x; c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot)\right).
$$

For simplicity, we will define the following notations for compensations

$$
\begin{aligned}
\Upsilon(\tau_i) &= X\left(\tau_i\right) + \frac{k_i\left(\tau_i\right)}{\eta_i\left(\tau_i\right)}, \\
\bar{\Upsilon}(\tau_i) &= \frac{q_i\left(\tau_i\right)}{h_i\left(\tau_i\right)}.
\end{aligned}
$$

Now, for $i = 1, 2$, let

$$
L_i(t, c_i(t)) = U\left(c_i(t)\right),
$$

be the utility function describing the wage-earner's i preferences regarding consumption at some instant of time $t \in [0, T]$,

$$
R(X(T)) = U(X(T)),
$$

be the utility function for the terminal wealth at retirement time T,

$$
\begin{aligned}
Y^{\Upsilon}(\tau_i, \Upsilon(\tau_i)) &= U\left(\Upsilon(\tau_i)\right), \\
Y^{\bar{\Upsilon}}(\tau_i, \bar{\Upsilon}(\tau_i)) &= U\left(\bar{\Upsilon}(\tau_i)\right),
\end{aligned}
$$

are the wage-earner's utility functions for the size of $\Upsilon(t)$ and $\bar{\Upsilon}(t)$,respectively, at time $t \in [0, T]$. For simplicity, we will assume $Y^{\Upsilon}(t, \cdot) = Y^{\bar{\Upsilon}}(t, \cdot) = Y(t, \cdot)$.

The following assumption allows us to apply the DPP and derive an explicit solution later on in this Thesis.

Assumption 4. *The utility functions*

$$L_i : [0;T] \times \mathbb{R}^+ \to \mathbb{R}^+,$$

$$Y^{\Upsilon} : [0;T] \times \mathbb{R}^+ \to \mathbb{R}^+,$$

and

$$Y^{\bar{\Upsilon}} : [0;T] \times \mathbb{R}^+ \to \mathbb{R}^+,$$

are twice differentiable, strictly increasing and strictly concave functions on their second variable, and

$$R : \mathbb{R}^+ \to \mathbb{R}^+$$

is a twice differentiable, strictly increasing and strictly concave function.

Let $\mathcal{A}_i(t,x)$ be defined as the set of all admissible strategies for control problem after first death

$$\pi_i(\cdot) := (c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot))$$

for the evolution of the wealth process with boundary condition $X(t) = x$.

For any given $\pi_i \in \mathcal{A}_i(t,x)$ equation (4.1) can be written as

$$\begin{aligned} \max \tilde{J}_i\left(t,x;\pi_i(\cdot)\right) := \mathbb{E}_t \Bigg[& \int_t^{\tau_i \wedge T} w_i e^{-\delta s} L_i(s, c_i(s)) \mathrm{d}s \\ & + w_3 \mathbf{1}_{\{\tau_i \leq T\}} e^{-\delta \tau_i} \Bigg(Y(\tau_i, \Upsilon(\tau_i)) + Y(\tau_i, \bar{\Upsilon}(\tau_i)) \Bigg) \\ & + w_4 \mathbf{1}_{\{\tau_i > T\}} e^{-\delta T} R(X^{\pi_i}_{t,x}(T)) \Bigg], \end{aligned} \tag{4.2}$$

where $X^{\pi_i}_{t,x}(s)$ is the solution of the stochastic differential equation (3.10). We note that $X^{\pi_i}_{t,x}(s) \geq 0$ is the wealth process which starts from x at time $t \leq s$ when $\pi_i \in \mathcal{A}_i(t,x)$ is being selected.

The transformation of the control problem (4.2) into a one with a fixed planning horizon is explained in the following lemma.

Lemma 4.1. *Assume that all previous Assumptions (1)-(4) are hold. If the random variables τ_1 and τ_2 are independent of the filtration $\mathbb{F}$, then*

$$\tilde{J}_i(t,x;\pi_i(\cdot)) = \mathbb{E}_t\Bigg[\int_t^T \bigg(w_i e^{-\delta s}(1-F_i(s,t))L_i(s,c_i(s)) + w_3 e^{-\delta\tau_i} f_i(s,t) \times \big(Y(s,\Upsilon(s)) + Y(s,\bar{\Upsilon}(s))\big)\bigg) ds + w_4 e^{-\delta T}(1-F_i(T,t))R(X(T)) \mid \mathcal{F}_t\Bigg].$$

Proof. From equation (4.2), we can rewrite the functional $\tilde{J}_i$ as

$$\tilde{J}_i(t,x;\pi(\cdot)) = \mathbb{E}_t\Bigg[1_{\{\tau_i\geq T\}}\int_t^T w_i e^{-\delta s}L_i(s,c_i(s))ds + 1_{\{\tau_i<T\}}\int_t^{\tau_i} w_i e^{-\delta s}L_i(s,c_i(s))ds + w_3\mathbf{1}_{\{\tau_i\leq T\}}e^{-\delta\tau_i} \times \big(Y(\tau_i,\Upsilon(\tau_i)) + Y(\tau_i,\bar{\Upsilon}(\tau_i))\big) + w_4\mathbf{1}_{\{\tau_i>T\}}e^{-\delta T}R(X(T)) \mid \tau_i > t, \mathcal{F}_t\Bigg].$$

Based on equations (3.5) and (3.6), the conditional probability density of the random variable τ_i is given by $f_i(u,t)$ and since τ_i is independent of the filtration $\mathbb{F}$ for $i = 1,2$, then we have

$$\begin{aligned}\tilde{J}_i(t,x;\pi_i(\cdot)) = \mathbb{E}_t\Bigg[&(1-F_i(T,t))\int_t^T w_i e^{-\delta s}L_i(s,c_i(s))ds \\ &+ \int_t^T f_i(u,t)\int_t^u w_i e^{-\delta s}L_i(s,c_i(s))dsdu \\ &+ \int_t^T f_i(u,t)w_3 e^{-\delta u} \times \big(Y(u,\Upsilon(u)) + Y(u,\bar{\Upsilon}(u))\big)\,du \\ &+ w_4(1-F_i(T,t))e^{-\delta T}R(X(T)) \mid \mathcal{F}_t\Bigg].\end{aligned} \tag{4.3}$$

By the Fubini-Tonelli Theorem 2.40, since

$$f_i(u,t)e^{-\delta s}L_i(s,c_i(s)) \geq 0,$$

it follows that, the order of integration can be interchanged, so

$$\begin{aligned}\int_t^T f_i(u,t)\int_t^u e^{-\delta s}L_i(s,c_i(s))dsdu &= \int_t^T\int_t^u f_i(u,t)e^{-\delta s}L_i(s,c_i(s))dsdu \\ &= \int_t^T\int_s^T f_i(u,t)e^{-\delta s}L_i(s,c_i(s))duds \\ &= \int_t^T\left(\int_s^T f_i(u,t)du\right)e^{-\delta s}L_i(s,c_i(s))ds \\ &= \int_t^T \left(F_i(T,t)-F_i(s,t)\right)e^{-\delta s}L_i(s,c_i(s))ds.\end{aligned} \tag{4.4}$$

Hence, by equation (4.3) and equation (4.4) we get

$$\begin{aligned}\tilde{J}_i(t,x;\pi_i(\cdot)) = \mathbb{E}_t\Bigg[\int_t^T \bigg(& w_i e^{-\delta s}(1-F_i(s,t))L_i(s,c_i(s)) \\ &+ w_3 e^{-\delta\tau_i} f_i(s,t)\times\left(Y(s,\Upsilon(s))+Y(s,\bar{\Upsilon}(s))\right)\bigg)ds \\ &+ w_4 e^{-\delta T}(1-F_i(T,t))R(X(T)) \mid \mathcal{F}_t\Bigg],\end{aligned}$$

which concludes the proof. □

Now, we introduce the following definition which will help us stating the next lemma.

Definition 4.2. *[44] Let $X(\cdot)$ is a continuous stochastic process defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$, then X has a Markcov property if for $0<t<s<T$ we have*

$$\mathbb{E}_t\left[X_s \mid \mathcal{F}_t\right] = \mathbb{E}_t\left[X_s \mid X_t\right].$$

In other words, a stochastic process has the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present values) depends only upon the present state, not on the sequence of events that preceded it.

Lemma 4.3. *(DPP) For $0\le t<s<T$, then the maximum expected utility $V_i(t,x)$ satisfies the recursive relation*

$$\begin{aligned}V_i(t,x) = \sup_{\pi_i\in\mathcal{A}_i(t,x)}\mathbb{E}_t\Bigg[& e^{-\int_t^s\lambda_i(v)dv}V_i\left(s,X_{t,x}^{\pi_i}(s)\right)+\int_t^s\bigg(w_i e^{-\delta u}(1-F_i(u,t))L_i(u,c_i(u)) \\ &+ w_3 e^{-\delta\tau_i}f_i(u,t)\times\left(Y(u,\Upsilon_{t,x}^{\pi_i}(u))+Y(u,\bar{\Upsilon}_{t,x}^{\pi_i}(u))\right)\bigg)du\;\bigg|\;\mathcal{F}_t\Bigg].\end{aligned} \tag{4.5}$$

Proof. For any $\pi_i \in \mathcal{A}_i(t,x)$ with the corresponding wealth $X^{\pi_i}_{t,x}(\cdot)$ and the corresponding total legacy $\Upsilon^{\pi_i}_{t,x}(u)$ and $\bar{\Upsilon}^{\pi_i}_{t,x}$. Lemma 4.1 gives that

$$\begin{aligned}\tilde{J}_i(t,x;\pi_i(\cdot)) = \mathbb{E}_t\Bigg[\int_t^T \Bigg(& w_i e^{-\delta u}(1-F_i(u,t))L_i(u,c_i(u)) \\ &+ w_3 e^{-\delta\tau_i} f_i(u,t) \times \big(Y(u,\Upsilon^{\pi_i}_{t,x}(u)) + Y(u,\bar{\Upsilon}^{\pi_i}_{t,x}(u))\big)\Bigg)du \\ &+ w_4 e^{-\delta T}(1-F_i(T,t))R(X(T)) \,\Bigg|\, \mathcal{F}_t\Bigg].\end{aligned}$$

For any $0 \le t < s < T$ the above equation becomes

$$\begin{aligned}\tilde{J}_i(t,x;\pi_i(\cdot)) = \mathbb{E}_t\Bigg[\int_s^T \Bigg(& w_i e^{-\delta u}(1-F_i(u,t))L_i(u,c_i(u)) \\ &+ w_3 e^{-\delta\tau_i} f_i(u,t) \times \big(Y(u,\Upsilon^{\pi_i}_{t,x}(u)) + Y(u,\bar{\Upsilon}^{\pi_i}_{t,x}(u))\big)\Bigg)du \\ &+ \int_t^s \Bigg(w_i e^{-\delta u}(1-F_i(u,t))L_i(u,c_i(u)) \\ &+ w_3 e^{-\delta\tau_i} f_i(u,t) \times \big(Y(u,\Upsilon^{\pi_i}_{t,x}(u)) + Y(u,\bar{\Upsilon}^{\pi_i}_{t,x}(u))\big)\Bigg)du \\ &+ w_4 e^{-\delta T}(1-F_i(T,t))R(X(T)) \,\Bigg|\, \mathcal{F}_t\Bigg].\end{aligned} \tag{4.6}$$

Note that from equation (3.4) and we can get that for $i = 1,2$,

$$\begin{aligned} f_i(u,t) &= \lambda_i(u) e^{-\int_t^u \lambda_i(v)dv} \\ &= e^{-\int_t^s \lambda_i(v)dv} \lambda_i(u) e^{-\int_s^u \lambda_i(v)dv} \\ &= e^{-\int_t^s \lambda_i(v)dv} f_i(u,s). \end{aligned}$$

Similarly,

$$1 - F_i(u,t) = e^{-\int_t^s \lambda_i(v)dv}(1 - F_i(u,s)).$$

After substituting the values of $f_i(u,t)$ and $1 - F_i(u,t)$ in equation (4.6), we get

$$
\begin{aligned}
\tilde{J}_i(t,x;\pi_i(\cdot)) = \mathbb{E}_t\Bigg[& e^{-\int_t^s \lambda_i(v)dv}\Bigg(\int_s^T \Bigg(w_i e^{-\delta u}(1-F_i(u,s))L_i(u,c_i(u)) \\
& + w_3 e^{-\delta\tau_i} f_i(u,s) \times \left(Y(u,\Upsilon_{t,x}^{\pi_i}(u)) + Y(u,\bar{\Upsilon}_{t,x}^{\pi_i}(u))\right)\Bigg)du \\
& + w_4 e^{-\delta T}(1-F_i(T,s))R(X_{t,x}^{\pi_i}(T))\Bigg) \\
& + \int_t^s \Bigg(w_i e^{-\delta u}(1-F_i(u,t))L_i(u,c_i(u)) \\
& + w_3 e^{-\delta\tau_i} f_i(u,t) \times \left(Y(u,\Upsilon_{t,x}^{\pi_i}(u)) + Y(u,\bar{\Upsilon}_{t,x}^{\pi_i}(u))\right)\Bigg)du \,\Bigg|\, \mathcal{F}_t\Bigg].
\end{aligned} \tag{4.7}
$$

Note that the stochastic process $X_{t,x}^{\pi_i}(\cdot)$ has the Markov's Property from Definition 4.2, in which case

$$
\begin{aligned}
\mathbb{E}_t\left[R(X_{t,x}^{\pi_i}(T)) \mid \mathcal{F}_t\right] &= \mathbb{E}_t\left[R(X_{t,x}^{\pi_i}(T)) \,|\mathcal{F}_s|\, \mathcal{F}_t\right] \\
&= \mathbb{E}_t\left[R(X_{s,X_{t,x}(s)}^{\pi_i}(T) \mid \mathcal{F}_t\right],
\end{aligned}
$$

and $\pi_i(\cdot) = (c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot))$ defined on interval $[s,T]$ is in $\mathcal{A}_i(s, X_{t,x}^{\pi_i}(s))$.

Hence, identity (4.7) becomes

$$
\begin{aligned}
\tilde{J}_i(t,x;\pi_i(\cdot)) = \mathbb{E}_t\Bigg[& e^{-\int_t^s \lambda_i(v)dv}\tilde{J}_i\left(s, X_{t,x}^{\pi_i}(s);\pi_i(\cdot)\right) \\
& + \int_t^s \Bigg(w_i e^{-\delta u}(1-F_i(u,t))L_i(u,c_i(u)) \\
& + w_3 e^{-\delta\tau_i} f_i(u,t) \times \left(Y(u,\Upsilon_{t,x}^{\pi_i}(u)) + Y(u,\bar{\Upsilon}_{t,x}^{\pi_i}(u))\right)\Bigg)du \,\Bigg|\, \mathcal{F}_t\Bigg] \\
\leq \mathbb{E}_t\Bigg[& e^{-\int_t^s \lambda_i(v)dv}V_i\left(s, X_{t,x}^{\pi_i}(s)\right) + \int_t^s \Bigg(w_i e^{-\delta u}(1-F_i(u,t))L_i(u,c_i(u)) \\
& + w_3 e^{-\delta\tau_i} f_i(u,t) \times \left(Y(u,\Upsilon_{t,x}^{\pi_i}(u)) + Y(u,\bar{\Upsilon}_{t,x}^{\pi_i}(u))\right)\Bigg)du \,\Bigg|\, \mathcal{F}_t\Bigg].
\end{aligned}
$$

Note that since $\pi_i(\cdot) = (c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot))$ is arbitrary, then it follows that

$$
\begin{aligned}
V_i(t,x) \leq \sup_{\pi_i \in \mathcal{A}_i(t,x)} \mathbb{E}_t\Bigg[& e^{-\int_t^s \lambda_i(v)dv}V_i\left(s, X_{t,x}^{\pi_i}(s)\right) + \int_t^s \Bigg(w_i e^{-\delta u}(1-F_i(u,t))L_i(u,c_i(u)) \\
& + w_3 e^{-\delta\tau_i} f_i(u,t) \times \left(Y(u,\Upsilon_{t,x}^{\pi_i}(u)) + Y(u,\bar{\Upsilon}_{t,x}^{\pi_i}(u))\right)\Bigg)du \,\Bigg|\, \mathcal{F}_t\Bigg].
\end{aligned} \tag{4.8}
$$

Thus, we conclude the first direction.

Conversely, given that $\pi_i(\cdot) = (c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot)) \in \mathcal{A}_i(t,x)$, for any $\epsilon > 0$ and $\omega \in \Omega$, and using the property of supremum, there exists

$$\mathcal{L}_i \equiv (c_{i,\omega,\epsilon}(\cdot), k_{i,\omega,\epsilon}(\cdot), q_{i,\omega,\epsilon}(\cdot), u_{i,\omega,\epsilon}(\cdot)) \in \mathcal{A}_i\left(s, X^{\pi_i}_{t,x}(s,\omega)\right),$$

where

$$\tilde{J}_i\left(s, X^{\pi_i}_{t,x}(s); \mathcal{L}_{i,\omega,\epsilon}(\cdot)\right) \geq V_i\left(s, X^{\pi_i}_{t,x}(s)\right) - \epsilon.$$

Let

$$\mathcal{L}^*_i(u) := \begin{cases} (c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot)), & \text{if } u \in [t,s], \\ \mathcal{L}_{i,\omega,\epsilon}(u), & \text{if } u \in [s,T]. \end{cases}$$

Notice that $X^{\mathcal{L}^*_i}_{t,x}(T) = X^{\mathcal{L}_{i,\omega,\epsilon}}_{s,X^{\pi_i}_{t,x}(s)}(T)$ a.s., then from the functional in equation (4.7), we get

$$V_i(t,x) \geq \tilde{J}_i\left(t, x; \mathcal{L}^*_i(\cdot)\right).$$

That is,

$$\begin{aligned}
&\mathbb{E}_t\Bigg[e^{-\int_t^s \lambda_i(v)dv}\Bigg(\int_s^T \Bigg(w_i e^{-\delta u}(1-F_i(u,s))L_i(u, c_{i,\omega,\epsilon}(u)) \\
&\quad + w_3 e^{-\delta\tau_i} f_i(u,s)\left(Y(u, \Upsilon^{\mathcal{L}_{i,\omega,\epsilon}}_{t,x}(u)) + Y(u, \bar{\Upsilon}^{\mathcal{L}_{i,\omega,\epsilon}}_{t,x}(u))\right)\Bigg)du \\
&\quad + w_4 e^{-\delta T}(1-F_i(T,s))R(X^{\mathcal{L}_{i,\omega,\epsilon}}_{s,X^{\pi_i}_{t,x}(s)}(T))\Bigg) \\
&\quad + \int_t^s \Bigg(w_i e^{-\delta u}(1-F_i(u,t))L_i(u, c_i(u)) \\
&\quad + w_3 e^{-\delta\tau_i} f_i(u,t)\left(Y(u, \Upsilon^{\pi_i}_{t,x}(u)) + Y(u, \bar{\Upsilon}^{\pi_i}_{t,x}(u))\right)\Bigg)du \,\Bigg|\, \mathcal{F}_t\Bigg] \\
\geq\; &\mathbb{E}_t\Bigg[e^{-\int_t^s \lambda_i(v)dv}\left(V_i\left(s, X^{\pi_i}_{t,x}(s)\right) - \epsilon\right) \\
&\quad + \int_t^s \Bigg(w_i e^{-\delta u}(1-F_i(u,t))L_i(u, c_i(u)) \\
&\quad + w_3 e^{-\delta\tau_i} f_i(u,t)\left(Y(u, \Upsilon^{\pi_i}_{t,x}(u)) + Y(u, \bar{\Upsilon}^{\pi_i}_{t,x}(u))\right)\Bigg)du \,\Bigg|\, \mathcal{F}_t\Bigg].
\end{aligned}$$

The above inequality holds for any $\pi_i(\cdot) = (c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot)) \in \mathcal{A}_i(t,x)$ and $\epsilon > 0$, which means

$$V_i(t,x) \geq \sup_{\pi_i \in \mathcal{A}_i(t,x)} \mathbb{E}_t \Bigg[e^{-\int_t^s \lambda_i(v)dv} V_i\left(s, X_{t,x}^{\pi_i}(s)\right) + \int_t^s \Bigg(w_i e^{-\delta u} F_i^+(u,t) L_i(u, c_i(u)) + w_3 e^{-\delta \tau_i} f_i(u,t) \times \left(Y(u, \Upsilon_{t,x}^{\pi_i}(u)) + Y(u, \bar{\Upsilon}_{t,x}^{\pi_i}(u)) \right) \Bigg) du \,\Bigg|\, \mathcal{F}_t \Bigg]. \tag{4.9}$$

Hence, from inequalities (4.8) and (4.9), we conclude the proof of Lemma 4.3. □

Dynamic programming principle helps us to write a second-order nonlinear partial differential equation whose solution is the value function of the optimal control problem under consideration.

4.2 Hamilton–Jacobi–Bellman equation (HJB)

We will use the *DPP* obtained in Lemma 4.3 to derived the following PDE, which is known as Hamilton–Jacobi–Bellman equation (HJB), and whose solution is the value function for the optimal control problem under consideration.

Theorem 4.4. *(HJB-Equation) Suppose that the maximum expected utility $V_i(t,x) \in C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$ for $i = 1, 2$. Then, $V_i(t,x)$ must satisfies the following HJB equation*

$$\begin{cases} V_{i,t}(t,x) - \lambda_i(t) V_i(t,x) + \sup_{(c_i,k_i,q_i,u_i)} \mathcal{H}_i(t,x;\pi_i(\cdot)) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}, \\ V_i(T,x) = w_4 e^{-\delta T} R(x), \quad x \in \mathbb{R}, \quad i = 1, 2. \end{cases}$$

Where the Hamilltonian $\mathcal{H}_i$ given by

$$\begin{aligned} \mathcal{H}_i(t,x;\pi_i(\cdot)) = &\Big[r(t)x + (\mu(t) - r(t))u_i(t) - c_i(t) - k_i(t) - q_i(t) + I_i(t) \Big] V_{i,x}(t,x) \\ &+ \frac{1}{2}\sigma^2(t) u_i^2(t) V_{i,xx}(t,x) + \lambda_i(t) w_3 e^{-\delta t} \left(Y\left(t, x + \frac{k_i(t)}{\eta_i(t)}\right) + Y\left(t, \frac{q_i(t)}{h_i(t)}\right) \right) \\ &+ w_i e^{-\delta t} L_i(t, c_i), \end{aligned}$$

$V_{i,t}$ and $V_{i,x}$ denote first-order partial derivatives with respect to t and x, respectively, and $V_{i,xx}$ denotes a second-order derivative with respect to x. Moreover,

$$\pi_i^*(\cdot) = (c_i^*(\cdot), k_i^*(\cdot), q_i^*(\cdot), u_i^*(\cdot)) \in \mathcal{A}_i(t,x),$$

whose wealth X^* is optimal if and if for $s \in [t, T]$ we have

$$V_{i,t}\left(s, X_i^*(s)\right) - \lambda_i(s) V_i\left(s, X_i^*(s)\right) + \mathcal{H}_i\left(s, X_i^*(s); \pi_i^*\right) = 0. \tag{4.10}$$

Proof. Apply $s = t + h$ in the DPP from equation(4.5). By Itô's formula from Theorem 2.56, we can get

$$\begin{aligned} V_i \quad & (t+h, X(t+h)) = V_i(t,x) + \int_t^{t+h} \Big(V_{i,t}(s, X(s)) + V_{i,x}(s, X(s)) \\ \times \quad & \Big[r(s) X_i(s) + (\mu(s) - r(s)) u_i(s) - c_i(s) - k_i(s) - q_i(s) + I_i(s) \Big] \\ + \quad & \frac{1}{2} V_{i,xx}(s, X(s)) \sigma^2(s) u_i^2(s) \Big) ds + \int_t^{t+h} V_{i,x}(s, X(s)) \sigma(s) u_i(s) dW(s). \end{aligned} \tag{4.11}$$

Note that by Taylor series expansion, and since h is small we can get that

$$e^{-\int_t^{t+h} \lambda_i(v) dv} = 1 - \lambda_i(h) + O(h^2), \tag{4.12}$$

where $O(h^2)$ represents an error of order two. Using equation (4.12) and Lemma 4.3 we get

$$\begin{aligned} 0 = \sup_{(c_i, k_i, q_i, u_i)} \mathbb{E}_t \Bigg[& \left(1 - \lambda_i(t) h + O(h^2)\right) V_i(t+h, X_i(t+h)) - V_i(t,x) \\ & + \int_t^{t+h} \Bigg(w_i e^{-\delta t} (1 - F_i(u,t)) L_i\left(t, c_i(u)\right) \\ & + w_3 e^{-\delta t} f_i(u,t) \left(Y\left(t, x + \frac{k_i\left(\tau_i\right)}{\eta_i\left(\tau_i\right)}\right) + Y\left(t, \frac{q_i\left(\tau_i\right)}{h_i\left(\tau_i\right)}\right) \right) \Bigg) du \mid \mathcal{F}_t \Bigg]. \end{aligned}$$

Now substitute equation (4.11) into the above equation, divide the result by h, and let h goes to zero to obtain

$$\begin{aligned} 0 = \sup_{(c_i, k_i, q_i, u_i)} \mathbb{E}_t \Bigg[& V_{i,t}(t,x) - \lambda_i(t) V_i(t,x) \\ & + \left[r(t) x + (\mu(t) - r(t)) u_i(t) - c_i(t) - k_i(t) - q_i(t) + I_i(t) \right] V_{i,x}(t,x) \\ & + \frac{1}{2} \sigma^2(t) u_i^2(t) V_{i,xx}(t,x) + w_i e^{-\delta t} L_i\left(t, c_i(u)\right) \\ & + \lambda_i(t) w_3 e^{-\delta t} \left(Y\left(t, x + \frac{k_i(t)}{\eta_i(t)}\right) + Y\left(t, \frac{q_i(t)}{h_i(t)}\right) \right) \Bigg]. \end{aligned}$$

Since $V_{i,t}(t,x) - \lambda_i(t) V_i(t,x)$ doesn't depend on π_i, it follows that the first part of HJB theorem holds.

Now we will prove the second part of the HJB theorem given in equation (4.10).
Let $(c_i(\cdot), k_i(\cdot), q_i(\cdot), u_i(\cdot)) \in \mathcal{A}_i(t,x)$ with the corresponding wealth X_i, applying Itô's formula to

$$e^{-\int_t^T \lambda_i(v)dv} V_i(s, X_i(s))$$

to obtain

$$\begin{aligned} V_i(t,x) = e^{-\int_t^T \lambda_i(v)dv} R(X_i(T)) - \int_t^T e^{-\int_t^u \lambda_i(v)dv} \Big(V_{i,t}(u, X_i(u)) \\ + V_{i,x}(u, X_i(u)) \Big(r(u)X_i(u) + (\mu(u) - r(u))u_i(u) - c_i(u) - k_i(u) - q_i(u) + I_i(u) \Big) \\ - \lambda_i(u)V_i(u, X_i(u)) + \frac{1}{2} V_{i,xx}(u, X_i(u))\sigma^2(u)u_i^2(u) \Big) du \\ - \int_t^T e^{-\int_t^u \lambda_i(v)dv} V_{i,x}(u, X_i(u))\sigma(u)u_i(u)dW(u). \end{aligned}$$

Now, take the expectation of the above equation to get

$$\begin{aligned} V_i(t,x) = \mathbb{E}_t \Big[e^{-\int_t^T \lambda_i(v)dv} R(X_i(T)) - \int_t^T e^{-\int_t^u \lambda_i(v)dv} \Big(V_{i,t}(u, X_i(u)) \\ + V_{i,x}(u, X_i(u)) \Big(r(u)X_i(u) + (\mu(u) - r(u))u_i(u) - c_i(u) - k_i(u) - q_i(u) + I_i(u) \Big) \\ - \lambda_i(u)V_i(u, X_i(u)) + \frac{1}{2} V_{i,xx}(u, X_i(u))\sigma^2(u)u_i^2(u) \Big) du \\ - \int_t^T e^{-\int_t^u \lambda_i(v)dv} V_{i,x}(u, X_i(u))\sigma(u)u_i(u)dW(u) \Big]. \end{aligned}$$

Using the property of linearity of expectation and substituting the value of the functional $\tilde{J}_i(t, x; c_i, k_i, q_i, u_i)$ from equation (4.1), we get

$$\begin{aligned} V_i(t,x) = \tilde{J}_i(t, x; c_i, k_i, q_i, u_i) - \mathbb{E}_t \Big[\int_t^T e^{-\int_t^u \lambda_i(v)dv} \Big(V_{i,t}(u, X_i(u)) \\ - \lambda_i(u)V_i(u, X_i(u)) + \mathcal{H}_i(u, X_i(u); c_i, k_i, q_i, u_i) \Big) du \Big]. \end{aligned} \tag{4.13}$$

Thus, from equation (4.13) we notice that (since the controls are arbitrary)

$$\begin{aligned} V_i(t,x) = \tilde{J}_i\left(t, x; c_i^*, k_i^*, q_i^*, u_i^*\right) - \mathbb{E}_t \Big[\int_t^T e^{-\int_t^u \lambda_i(v)dv} \Big(V_{i,t}\left(u, X_i^*(u)\right) \\ -\lambda_i(u)V_i\left(u, X_i^*(u)\right) + \mathcal{H}_i\left(u, X_i^*(u); c_i^*, k_i^*, q_i^*, u_i^*\right) \Big) du \Big]. \end{aligned}$$

Since $V_i(t,x) - \tilde{J}_i\left(t, x; c_i^*, k_i^*, q_i^*, u_i^*\right) \geq 0$, we conclude that

$$V_{i,t}\left(s, X_i^*(s)\right) - \lambda_i(s)V_i\left(s, X_i^*(s)\right) + \mathcal{H}_i\left(s, X_i^*(s); \pi_i^*\right) \leq 0. \tag{4.14}$$

Now to prove the other direction, since $V_i(t,x)$ is the maximum expected value, it follows that from equation (4.13)

$$\begin{aligned} V_i(t,x) \geq \tilde{J}_i(t,x;c_i,k_i,q_i,u_i) - \mathbb{E}_t \bigg[\int_t^T e^{-\int_t^u \lambda_i(v)dv} \Big(V_{i,t}(u,X_i(u)) \\ - \lambda_i(u)V_i(u,X_i(u)) + \sup_{(c_i,k_i,q_i,u_i)\in\mathcal{A}_i(t,x)} \mathcal{H}_i(u,X_i(u);c_i,k_i,q_i,u_i) \Big) du \bigg]. \end{aligned} \tag{4.15}$$

From inequality (4.15), we obtain

$$\begin{aligned} V_i(t,x) \geq \tilde{J}_i\left(t,x;c_i^*,k_i^*,q_i^*,u_i^*\right) - \mathbb{E}_t \bigg[\int_t^T e^{-\int_t^u \lambda_i(v)dv} \Big(V_{i,t}\left(u,X_i^*(u)\right) \\ - \lambda_i(u)V_i\left(u,X_i^*(u)\right) + \mathcal{H}_i\left(u,X_i^*(u);c_i^*,k_i^*,q_i^*,u_i^*\right) \Big) du \bigg]. \end{aligned}$$

Since $V_i(t,x)$ is the maximum expected value, that is

$$V_i(t,x) = \tilde{J}_i\left(t,x,c_i^*,k_i^*,q_i^*,u_i^*\right),$$

it follows that

$$V_{i,t}\left(s,X_i^*(s)\right) - \lambda_i(s)V_i\left(s,X_i^*(s)\right) + \mathcal{H}_i\left(s,X_i^*(s);\pi_i^*\right) \geq 0. \tag{4.16}$$

Hence, combining inequalities (4.14) and (4.16) we conclude the result. This completes the proof. □

4.3 Optimal strategies in terms of the value function

In this section, we will find the optimal strategies such as the optimal consumption $c_i^*(\cdot)$, optimal insurance premium $k_i^*(\cdot)$, optimal welfare policy $q_i^*(\cdot)$, and optimal portfolio $u_i^*(\cdot)$ in terms of the value function $V_i(t,x)$ for the wage-earner i, where $i=1,2$.

Based on Theorem 4.4 we will determine the optimal strategies $c_i^*(\cdot)$, $k_i^*(\cdot)$, $q_i^*(\cdot)$ and $u_i^*(\cdot)$ in terms of the value function $V_{i,x}(t;x)$ in the next result. To proceed we first introduce some properties for the utility functions $L_i(t;\cdot)$ and $Y(t;\cdot)$. From Assumption 4 we have that $L_i(t,\cdot)$ and $Y(t,\cdot)$ are strictly concave with respect to their second arguments. Hence, $L_{i,x}(t,\cdot)$ and $Y_x(t,\cdot)$ must be invertible for each $t\in[0,T]$, where $L_{i,x}(t,\cdot)$ and $Y_x(t,\cdot)$ are the derivatives of $L_i(t,\cdot)$ and $Y(t,\cdot)$ with respect to their second arguments, respectively. Let

$$N_1 : [0,T]\times\mathbb{R}_0^+ \to \mathbb{R}_0^+,$$

and

$$N_2 : [0,T] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+,$$

to be the unique inverse functions such that

$$\begin{aligned} N_1(t, L_{i,x}(t,x)) = x \quad &\text{and} \quad L_{i,x}(t, N_1(t,x)) = x, \\ N_2(t, Y_x(t,x)) = x \quad &\text{and} \quad Y_x(t, N_2(t,x)) = x, \end{aligned} \tag{4.17}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}_0^+$.

Next theorem give us the formula of the optimal strategies that we are looking for.

Theorem 4.5. *Suppose that the Assumptions 1-4 are hold and that the value function $V_i \in C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$. Then the Hamilltonian $\mathcal{H}_i$ has a unique maximum $\pi_i^* = (c_i^*(\cdot), k_i^*(\cdot), q_i^*(\cdot), u_i^*(\cdot)) \in \mathcal{A}_i(t,x)$, and the optimal strategies are given by*

$$\begin{aligned} c_i^*(t,x) &= N_1\left(t, \frac{V_{i,x}(t,x)}{w_i e^{-\delta t}}\right), \\ k_i^*(t,x) &= \left[N_2\left(t, \frac{\eta_i(t) V_{i,x}(t,x)}{\lambda_i(t) w_3 e^{-\delta t}}\right) - x\right]\eta_i(t), \\ q_i^*(t,x) &= N_2\left(t, \frac{h_i(t) V_{i,x}(t,x)}{\lambda_i(t) w_3 e^{-\delta t}}\right) h_i(t), \\ u_i^*(t,x) &= -\frac{(\mu(t) - r(t)) V_{i,x}(t,x)}{V_{i,xx}(t,x)\sigma^2(t)}, \end{aligned}$$

where N_1 and N_2 are as given in (4.17), respectively.

Proof. An optimal admissible strategy $(c_i^*, k_i^*, q_i^*, u_i^*) \in \mathcal{A}_i(t,x)$ whose wealth process X_i^* is satisfies identity (4.10) of Theorem 4.4.

Hence, it is enough to consider the following four independent conditions in maximizing $\mathcal{H}_i$

$$\begin{aligned} &\sup_{(c_i,k_i,q_i,u_i)\in(\mathbb{R}^+)^4} \mathcal{H}_i(t,x;c_i,k_i,q_i,u_i) = \sup_{c_i\in\mathbb{R}^+}\left\{w_i e^{-\delta t} L_i(t,c_i) - c_i(t)V_{i,x}(t,x)\right\} \\ &+ \sup_{k_i\in\mathbb{R}^+}\left\{\lambda_i(t) w_3 e^{-\delta t} Y\left(t, x + \frac{k_i(t)}{\eta_i(t)}\right) - k_i(t)V_{i,x}(t,x)\right\} \\ &+ \sup_{q_i\in\mathbb{R}^+}\left\{\lambda_i(t) w_3 e^{-\delta t} Y\left(t, \frac{q_i(t)}{h_i(t)}\right) - q_i(t)V_{i,x}(t,x)\right\} \\ &+ \sup_{u_i\in\mathbb{R}^+}\left\{\frac{1}{2}\sigma^2(t)u_i^2 V_{i,xx}(t,x) + (\mu(t) - r(t))u_i(t)V_{i,x}(t,x)\right\} \\ &+ (r(t)x + I_i(t))\, V_{i,x}(t,x). \end{aligned} \tag{4.18}$$

We derive $\mathcal{H}_i$ with respect the variables c_i^*, k_i^*, q_i^* and u_i^* as they follow non constrained maximization problems, respectively, we obtain the following first order conditions

$$\begin{aligned} w_i e^{-\delta t} L_{c_i^*}\left(t, c_i^*\right) - V_{i,x}(t,x) = 0, \\ \frac{\lambda_i(t)}{\eta_i(t)} w_3 e^{-\delta t} Y_x\left(t, x + \frac{k_i^*(t)}{\eta_i(t)}\right) - V_{i,x}(t,x) = 0, \\ \frac{\lambda_i(t)}{h_i(t)} w_3 e^{-\delta t} Y_x\left(t, \frac{q_i^*(t)}{h_i(t)}\right) - V_{i,x}(t,x) = 0, \\ \sigma^2(t) u_i^*(t,x) V_{i,xx}(t,x) + (\mu(t) - r(t)) V_{i,x}(t,x) = 0. \end{aligned} \tag{4.19}$$

Using the inverse functions stated in (4.17), we solve equations (4.19) for the corresponding control variables as

$$\begin{aligned} c_i^*(t,x) &= N_1\left(t, \frac{V_{i,x}(t,x)}{w_i e^{-\delta t}}\right), \\ k_i^*(t,x) &= \left[N_2\left(t, \frac{\eta_i(t) V_{i,x}\left(t,x\right)}{\lambda_i(t) w_3 e^{-\delta t}}\right) - x\right] \eta_i\left(t\right), \\ q_i^*(t,x) &= N_2\left(t, \frac{h_i\left(t\right) V_{i,x}\left(t,x\right)}{\lambda_i(t) w_3 e^{-\delta t}}\right) h_i\left(t\right), \\ u_i^*(t,x) &= -\frac{(\mu(t) - r(t)) V_{i,x}(t,x)}{V_{i,xx}(t,x)\sigma^2(t)}. \end{aligned}$$

We evaluate the second derivative to each one of the variables to get

$$\begin{aligned} \mathcal{H}_{i,c_i c_i}(t,x;\pi_i^*) &= w_i e^{-\delta t} L_{c_i c_i}(t, c_i^*), \\ \mathcal{H}_{i,k_i k_i}(t,x;\pi_i^*) &= \frac{\lambda_i(t)}{\eta_i^2(t)} w_3 e^{-\delta t} Y_{k_i k_i}\left(t, x + \frac{k_i^*(t)}{\eta_i(t)}\right), \\ \mathcal{H}_{i,q_i q_i}(t,x;\pi_i^*) &= \frac{\lambda_i(t)}{h_i^2(t)} w_3 e^{-\delta t} Y_{q_i q_i}\left(t, \frac{q_i^*(t)}{h_i(t)}\right), \\ \mathcal{H}_{i,u_i u_i}(t,x;\pi_i^*) &= V_{i,xx}(t,x)\sigma^2(t). \end{aligned}$$

Using the Assumption 4 about the strict concavity of the functions L_i and Y with respect to their second variables, we conclude that the optimal strategies c_i^*, k_i^* and q_i^* are optimal. Now we will show that $\mathcal{H}_{i,u_i u_i}(t,x;v_i^*)$ is negative. if $V_{i,xx}(t,x) > 0$, then from the HJB equation $\mathcal{H}_i$ has no upper bound and hence, either $V_{i,t}(t,x) = \infty$ or $V_i(t,x) = \infty$, which contradicts the assumption about smoothness of V_i. Therefore, $V_{i,xx}(t,x) < 0$ and so $\mathcal{H}_{i,u_i u_i}$ is negative. Hence, $\mathcal{H}_i$ has a unique regular interior maximum and we conclude the proof. □

4.4 The power utility function

In this section, we will assume the following power utility functions for the wage-earner i, $i = 1, 2,$:

$$
\begin{aligned}
L_i(t, c_i) &= \frac{(c_i)^\gamma}{\gamma}, \\
Y\left(t, x + \frac{k_i(t)}{\eta_i(t)}\right) &= \frac{\left(x + \frac{k_i(t)}{\eta_i(t)}\right)^\gamma}{\gamma}, \\
Y\left(t, \frac{q_i(t)}{h_i(t)}\right) &= \frac{\left(\frac{q_i(t)}{h_i(t)}\right)^\gamma}{\gamma},
\end{aligned}
\tag{4.20}
$$

where $\gamma < 1$ denotes the risk parameter and not equal to zero. We will use these utilities to derive an explicit solutions for the problem under consideration.

4.5 Explicit solution

In the following theorem, we will derive an explicit solution for the optimal controls by using the optimal strategies obtained in equation (4.19) and applying the power utilities, we get the following explicit theorem after the first death for the wage-earner i.

Theorem 4.6. *Given the power utility functions in equation* (4.20)*. Assume the value function corresponding to wage-earner i after T_1 is given by*

$$V_i(t, x) = e^{-\delta t} \frac{a_i(t)}{\gamma} \left(x + b_i(t)\right)^\gamma,$$

where $T_1 = \tau_{3-i}$, $i = 1, 2$.

Then the optimal strategies are given by

$$
\begin{cases}
c_i(t, x) = w_i^{-\frac{1}{\gamma-1}} \frac{1}{l_i(t)} \left(x + b_i(t)\right), \\
u_i(t, x) = -\frac{\mu(t) - r(t)}{\sigma^2(t)(\gamma - 1)} \left(x + b_i(t)\right), \\
k_i(t, x) = \eta_i(t) \left(\left(\frac{\eta_i(t)}{\lambda_i(t) w_3}\right)^{\frac{1}{\gamma-1}} \frac{1}{l_i(t)} \left(x + b_i(t)\right) - x\right), \\
q_i(t, x) = h_i(t) \left(\frac{h_i(t)}{\lambda_i(t) w_3}\right)^{\frac{1}{\gamma-1}} \frac{1}{l_i(t)} \left(x + b_i(t)\right),
\end{cases}
$$

where

$$\begin{aligned}
a_i(t) =& l_i^{1-\gamma}(t),\\
b_i(t) =& \int_t^T I_i(s) e^{-\int_t^s (r(z)+\eta_i(z))dz} \mathrm{d}s,\\
l_i(t) =& w_4^{\frac{1}{1-\gamma}} e^{-\int_t^T h_i(s)ds} + \int_t^T e^{-\int_t^s h_i(z)\mathrm{d}z} g_i(s)\mathrm{d}s,\\
h_i(t) =& \frac{1}{1-\gamma}\left(\delta + \lambda_i(t) - \gamma\left(r(t)+\eta_i(t)\right) - \frac{\gamma}{2}\frac{(\mu(t)-r(t))^2}{\sigma^2(t)(1-\gamma)}\right),\\
g_i(t) =& w_i^{-\frac{1}{\gamma-1}} + \left(\frac{\eta_i^{\frac{\gamma}{\gamma-1}} + h_i^{\frac{\gamma}{\gamma-1}}}{(\lambda_i(t)w_3)^{\frac{1}{\gamma-1}}}\right).
\end{aligned}$$

Proof. We start the proof by noticing from condition (4.19), we get

$$V_{i,x}(t,x) = w_i e^{-\delta t} L_{c_i}\left(t, c_i^*\right).$$

Substitute the value of $L_{c_i}\left(t, c_i^*\right)$ we get

$$V_{i,x}(t,x) = w_i e^{-\delta t}\left(c_i^*\right)^{\gamma-1}.$$

After rearrange the above equation for $c_i^*(t,x)$ we get

$$c_i^*(t,x) = \left(\frac{V_{i,x}(t,x)e^{\delta t}}{w_i}\right)^{\frac{1}{\gamma-1}}. \tag{4.21}$$

Substituting the value of Y_x in equation (4.19) to get

$$V_{i,x}(t,x) = \frac{\lambda_i(t)}{\eta_i(t)} w_3 e^{-\delta t}\left(x + \frac{k_i^*(t)}{\eta_i(t)}\right)^{\gamma-1}. \tag{4.22}$$

Rearranging equation (4.22) we get

$$k_i^*(t,x) = \left(\left(\frac{\eta_i(t)}{\lambda_i(t)w_3} V_{i,x}(t,x)e^{\delta t}\right)^{\frac{1}{\gamma-1}} - x\right)\eta_i(t). \tag{4.23}$$

Similarly, to find the optimal control $q_i^*(t,x)$. We use again equation (4.19), we obtain

$$V_{i,x}(t,x) = \frac{\lambda_i(t)}{h_i(t)} w_3 e^{-\delta t}\left(\frac{q_i^*(t)}{h_i(t)}\right)^{\gamma-1}. \tag{4.24}$$

Rearranging equation (4.24), we get

$$q_i^*(t,x) = \left(\frac{h_i(t)}{\lambda_i(t)w_3} V_{i,x}(t,x)e^{\delta t}\right)^{\frac{1}{\gamma-1}} h_i(t). \tag{4.25}$$

Similarly, from equation (4.19), we have that

$$u_i^*(t,x) = -\frac{(\mu(t)-r(t))V_{i,x}(t,x)}{\sigma^2(t)V_{i,xx}(t,x)}. \tag{4.26}$$

Now we are going to find the solution for the HJB equation. We will substitute the optimal strategies c_i^*, k_i^*, q_i^* and u_i^* from equations (4.21), (4.23), (4.25) and (4.26), respectively, in the HJB equation, but we will do that in a separated steps as the following

Step 1: Substitute the values of c_i^* from equation (4.21) and $L_i(t,c_i^*)$ from equation (4.20) in equation (4.18), to get

$$\begin{aligned}
&\sup_{c_i\in\mathbb{R}^+}\left\{w_i e^{-\delta t}L_i\left(t,c_i\right) - c_i(t)V_{i,x}(t,x)\right\} \\
=&\frac{w_i e^{-\delta t}}{\gamma}\left(\frac{V_{i,x}(t,x)e^{\delta t}}{w_i}\right)^{\frac{\gamma}{\gamma-1}} - \left(\frac{V_{i,x}(t,x)e^{\delta t}}{w_i}\right)^{\frac{1}{\gamma-1}}V_{i,x}(t,x).
\end{aligned}$$

Rearrange the above terms, we get

$$\sup_{c_i\in\mathbb{R}^+}\left\{w_i e^{-\delta t}L_i\left(t,c_i\right) - c_i(t)V_{i,x}(t,x)\right\} = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{e^{\delta t}}{w_i}\right)^{\frac{1}{\gamma-1}}\left(V_{i,x}(t,x)\right)^{\frac{\gamma}{\gamma-1}}. \tag{4.27}$$

Step 2: Substitute the value of u_i^* from equation (4.26) in equation (4.18), to get

$$\begin{aligned}
&\sup_{u_i\in\mathbb{R}^+}\left\{\frac{1}{2}\sigma^2(t)u_i^2V_{i,xx}(t,x) + (\mu(t)-r(t))u_i(t)V_{i,x}(t,x)\right\} \\
=&\frac{\sigma^2(t)}{2}\left(\frac{-(\mu(t)-r(t))V_{i,x}(t,x)}{\sigma^2(t)V_{i,xx}(t,x)}\right)^2 V_{i,xx}(t,x) \\
&+(\mu(t)-r(t))\left(\frac{-(\mu(t)-r(t))\left(V_{i,x}(t,x)\right)^2}{\sigma^2(t)V_{i,xx}(t,x)}\right).
\end{aligned}$$

Rearrange the above terms, we obtain

$$\begin{aligned}
&\sup_{u_i\in\mathbb{R}^+}\left\{\frac{1}{2}\sigma^2(t)u_i^2V_{i,xx}(t,x) + (\mu(t)-r(t))u_i(t)V_{i,x}(t,x)\right\} \\
&= -\frac{(\mu(t)-r(t))^2\left(V_{i,x}(t,x)\right)^2}{2\sigma^2(t)V_{i,xx}(t,x)}.
\end{aligned} \tag{4.28}$$

Step 3: Substitute the value of Y from equation (4.20) in equation (4.18), we get

$$\begin{aligned}&\sup_{k_i\in\mathbb{R}^+}\left\{\lambda_i(t)w_3e^{-\delta t}Y\left(t,x+\frac{k_i(t)}{\eta_i(t)}\right)-k_i(t)V_{i,x}(t,x)\right\}\\=&\frac{\lambda_i(t)w_3e^{-\delta t}}{\gamma}\left(x+\frac{k_i^*(t)}{\eta_i(t)}\right)^{\gamma}-k_i^*(t)V_{i,x}(t,x).\end{aligned}$$

Substitute the value of k_i^* from equation (4.23) in equation (4.18) to obtain

$$\begin{aligned}&\sup_{k_i\in\mathbb{R}^+}\left\{\lambda_i(t)w_3e^{-\delta t}Y\left(t,x+\frac{k_i(t)}{\eta_i(t)}\right)-k_i(t)V_{i,x}(t,x)\right\}\\=&\frac{\lambda_i(t)w_3e^{-\delta t}}{\gamma}\left(x+\frac{\left(\left(\frac{\eta_i(t)}{\lambda_i(t)w_3}V_{i,x}(t,x)e^{\delta t}\right)^{\frac{1}{\gamma-1}}-x\right)\eta_i(t)}{\eta_i(t)}\right)^{\gamma}\\-&\left(\left(\frac{\eta_i(t)}{\lambda_i(t)w_3}V_{i,x}(t,x)e^{\delta t}\right)^{\frac{1}{\gamma-1}}-x\right)\eta_i(t)V_{i,x}(t,x).\end{aligned}$$

That is,

$$\begin{aligned}&\sup_{k_i\in\mathbb{R}^+}\left\{\lambda_i(t)w_3e^{-\delta t}Y\left(t,x+\frac{k_i(t)}{\eta_i(t)}\right)-k_i(t)V_{i,x}(t,x)\right\}\\=&\frac{\lambda_i(t)w_3e^{-\delta t}}{\gamma}\left(\frac{\eta_i(t)}{\lambda_i(t)w_3}V_{i,x}(t,x)e^{\delta t}\right)^{\frac{\gamma}{\gamma-1}}\\-&\left(\frac{\eta_i(t)}{\lambda_i(t)w_3}V_{i,x}(t,x)e^{\delta t}\right)^{\frac{1}{\gamma-1}}\eta_i(t)V_{i,x}(t,x)+x\eta_i(t)V_{i,x}(t,x).\end{aligned}$$

Rearrange the terms to get

$$\begin{aligned}&\sup_{k_i\in\mathbb{R}^+}\left\{\lambda_i(t)w_3e^{-\delta t}Y\left(t,x+\frac{k_i(t)}{\eta_i(t)}\right)-k_i(t)V_{i,x}(t,x)\right\}\\=&x\eta_i(t)V_{i,x}(t,x)+\left(\frac{1-\gamma}{\gamma}\right)e^{\frac{\delta t}{\gamma-1}}\left(\frac{\eta_i(t)}{(\lambda_i(t)w_3)^{\frac{1}{\gamma}}}V_{i,x}(t,x)\right)^{\frac{\gamma}{\gamma-1}}.\end{aligned}\tag{4.29}$$

Step 4: Substitute the value of Y from equation (4.20) in equation (4.18) we get that

$$\begin{aligned}&\sup_{q_i\in\mathbb{R}^+}\left\{\lambda_i(t)w_3e^{-\delta t}Y\left(t,\frac{q_i(t)}{h_i(t)}\right)-q_i(t)V_{i,x}(t,x)\right\}\\=&\frac{\lambda_i(t)w_3e^{-\delta t}}{\gamma}\left(\frac{q_i^*(t)}{h_i(t)}\right)^{\gamma}-q_i^*(t)V_{i,x}(t,x).\end{aligned}$$

Substitute the value of q_i^* from equation (4.25) in equation (4.18) to get

$$
\begin{aligned}
&\sup_{q_i\in\mathbb{R}^+}\left\{\lambda_i(t)w_3e^{-\delta t}Y\left(t,\frac{q_i(t)}{h_i(t)}\right)-q_i(t)V_{i,x}(t,x)\right\}\\
=&\frac{\lambda_i(t)w_3e^{-\delta t}}{\gamma}\left(\frac{\left(\frac{h_i(t)}{\lambda_i(t)w_3}V_{i,x}(t,x)e^{\delta t}\right)^{\frac{1}{\gamma-1}}h_i(t)}{h_i(t)}\right)^{\gamma}\\
-&\left(\frac{h_i(t)}{\lambda_i(t)w_3}V_{i,x}(t,x)e^{\delta t}\right)^{\frac{1}{\gamma-1}}h_i(t)V_{i,x}(t,x).
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\sup_{q_i\in\mathbb{R}^+}\left\{\lambda_i(t)w_3e^{-\delta t}Y\left(t,\frac{q_i(t)}{h_i(t)}\right)-q_i(t)V_{i,x}(t,x)\right\}\\
=&\frac{\lambda_i(t)w_3e^{-\delta t}}{\gamma}\left(\frac{h_i(t)}{\lambda_i(t)w_3}V_{i,x}(t,x)e^{\delta t}\right)^{\frac{\gamma}{\gamma-1}}\\
-&\left(\frac{h_i(t)}{\lambda_i(t)w_3}V_{i,x}(t,x)e^{\delta t}\right)^{\frac{1}{\gamma-1}}h_i(t)V_{i,x}(t,x).
\end{aligned}
$$

Rearrange the terms to get

$$
\begin{aligned}
&\sup_{q_i\in\mathbb{R}^+}\left\{\lambda_i(t)w_3e^{-\delta t}Y\left(t,\frac{q_i(t)}{h_i(t)}\right)-q_i(t)V_{i,x}(t,x)\right\}\\
=&\left(\frac{1-\gamma}{\gamma}\right)e^{\frac{\delta t}{\gamma-1}}\left(\frac{h_i(t)}{(\lambda_i(t)w_3)^{\frac{1}{\gamma}}}V_{i,x}(t,x)\right)^{\frac{\gamma}{\gamma-1}}.
\end{aligned}
\tag{4.30}
$$

After substitute the values from equations (4.27), (4.28), (4.29) and (4.30) in equation (4.18) these computations lead to

$$
\begin{aligned}
\sup_{\pi_i\in(\mathbb{R}^+)^4}\mathcal{H}_i(t,x,\pi_i) &= \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{e^{\delta t}}{w_i}\right)^{\frac{1}{\gamma-1}}\Big(V_{i,x}(t,x)\Big)^{\frac{\gamma}{\gamma-1}}+\left(r(t)x+I_i(t)\right)V_{i,x}(t,x)\\
&+\left(\frac{1-\gamma}{\gamma}\right)e^{\frac{\delta t}{\gamma-1}}\left(\frac{\eta_i(t)}{(\lambda_i(t)w_3)^{\frac{1}{\gamma}}}V_{i,x}(t,x)\right)^{\frac{\gamma}{\gamma-1}}+x\eta_i(t)V_{i,x}(t,x)\\
&+\left(\frac{1-\gamma}{\gamma}\right)e^{\frac{\delta t}{\gamma-1}}\left(\frac{h_i(t)}{(\lambda_i(t)w_3)^{\frac{1}{\gamma}}}V_{i,x}(t,x)\right)^{\frac{\gamma}{\gamma-1}}\\
&-\frac{(\mu(t)-r(t))^2\left(V_{i,x}(t,x)\right)^2}{2\sigma^2(t)V_{i,xx}(t,x)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sup_{\pi_i\in(\mathbb{R}^+)^4}\mathcal{H}_i(t,x,\pi_i) &= \left(\frac{1-\gamma}{\gamma}\right)e^{\frac{\delta t}{\gamma-1}}\Big(V_{i,x}(t,x)\Big)^{\frac{\gamma}{\gamma-1}}\left(w_i^{-\frac{1}{\gamma-1}}+\left(\frac{\eta_i^{\frac{\gamma}{\gamma-1}}+h_i^{\frac{\gamma}{\gamma-1}}}{(\lambda_i(t)w_3)^{\frac{1}{\gamma-1}}}\right)\right)\\
&+\Big(r(t)x+\eta_i(t)x+I_i(t)\Big)V_{i,x}(t,x)-\frac{(\mu(t)-r(t))^2\left(V_{i,x}(t,x)\right)^2}{2\sigma^2(t)V_{i,xx}(t,x)}.
\end{aligned}
$$

Now substitute in the HJB equation from Theorem 4.4 to obtain

$$\begin{aligned}
0 &= V_{i,t}(t,x) - \lambda_i(t)V_i(t,x) \\
&+ \left(\frac{1-\gamma}{\gamma}\right) e^{\frac{\delta t}{\gamma-1}} \Big(V_{i,x}(t,x)\Big)^{\frac{\gamma}{\gamma-1}} \left(w_i^{-\frac{1}{\gamma-1}} + \left(\frac{\eta_i^{\frac{\gamma}{\gamma-1}} + h_i^{\frac{\gamma}{\gamma-1}}}{(\lambda_i(t)w_3)^{\frac{1}{\gamma-1}}}\right)\right) \\
&+ \Big(r(t)x + \eta_i(t)x + I_i(t)\Big)V_{i,x}(t,x) - \frac{(\mu(t)-r(t))^2\,(V_{i,x}(t,x))^2}{2\sigma^2(t)V_{i,xx}(t,x)}.
\end{aligned}$$

To make this simple, consider

$$g_i(t) = w_i^{-\frac{1}{\gamma-1}} + \left(\frac{\eta_i^{\frac{\gamma}{\gamma-1}} + h_i^{\frac{\gamma}{\gamma-1}}}{(\lambda_i(t)w_3)^{\frac{1}{\gamma-1}}}\right).$$

Then the HJB equation simplifies to

$$\begin{aligned}
0 &= V_{i,t}(t,x) - \lambda_i(t)V_i(t,x) + \left(\frac{1-\gamma}{\gamma}\right) e^{\frac{\delta t}{\gamma-1}} \Big(V_{i,x}(t,x)\Big)^{\frac{\gamma}{\gamma-1}} g_i(t) \\
&+ \Big(r(t)x + \eta_i(t)x + I_i(t)\Big)V_{i,x}(t,x) - \frac{(\mu(t)-r(t))^2\,(V_{i,x}(t,x))^2}{2\sigma^2(t)V_{i,xx}(t,x)},
\end{aligned} \tag{4.31}$$

with the terminal condition

$$V_i(T,x) = w_4 e^{-\delta T} R(x).$$

Now to solve equation (4.31), recall the ansatz function from the statement of this theorem

$$V_i(t,x) = e^{-\delta t}\frac{a_i(t)}{\gamma}\,(x+b_i(t))^{\gamma}.$$

Compute the partial derivatives $V_{i,t}$, $V_{i,x}$ and $V_{i,xx}$

$$\begin{aligned}
V_{i,t}(t,x) &= e^{-\delta t}a_i(t)\Big(x+b_i(t)\Big)^{\gamma-1}\frac{db_i(t)}{dt} + \frac{\Big(x+b_i(t)\Big)^{\gamma} e^{-\delta t}}{\gamma}\left(\frac{da_i(t)}{dt} - \delta a_i(t)\right), \\
V_{i,x}(t,x) &= e^{-\delta t}a_i(t)\,(x+b_i(t))^{\gamma-1}, \\
V_{i,xx}(t,x) &= (\gamma-1)e^{-\delta t}a_i(t)\,(x+b_i(t))^{\gamma-2}.
\end{aligned} \tag{4.32}$$

Substitute the above partial derivatives in equation (4.31) to get that

$$
\begin{aligned}
0 &= e^{-\delta t}a_i(t)\Big(x+b_i(t)\Big)^{\gamma-1}\frac{db_i(t)}{dt}+\frac{\Big(x+b_i(t)\Big)^{\gamma}e^{-\delta t}}{\gamma}\Bigg(\frac{da_i(t)}{dt}-\delta a_i(t)\Bigg)\\
&- \lambda_i(t)e^{-\delta t}\frac{a_i(t)}{\gamma}\left(x+b_i(t)\right)^{\gamma}+\left(\frac{1-\gamma}{\gamma}\right)e^{\frac{\delta t}{\gamma-1}}\Big(e^{-\delta t}a_i(t)\left(x+b_i(t)\right)^{\gamma-1}\Big)^{\frac{\gamma}{\gamma-1}}g_i(t)\\
&+ \Big(r(t)x+\eta_i(t)x+I_i(t)\Big)e^{-\delta t}a_i(t)\left(x+b_i(t)\right)^{\gamma-1}\\
&- \frac{(\mu(t)-r(t))^2\Big(e^{-\delta t}a_i(t)\left(x+b_i(t)\right)^{\gamma-1}\Big)^2}{2\sigma^2(t)(\gamma-1)e^{-\delta t}a_i(t)\left(x+b_i(t)\right)^{\gamma-2}}.
\end{aligned}
$$

Divide the last equation by $(x+b_i(t))^{\gamma}$ we get

$$
\begin{aligned}
0 &= \frac{e^{-\delta t}a_i(t)}{(x+b_i(t))}\frac{db_i(t)}{dt}+\frac{e^{-\delta t}}{\gamma}\Bigg(\frac{da_i(t)}{dt}-\delta a_i(t)\Bigg)\\
&- \lambda_i(t)e^{-\delta t}\frac{a_i(t)}{\gamma}+\left(\frac{1-\gamma}{\gamma}\right)e^{-\delta t}\Big(a_i(t)\Big)^{\frac{\gamma}{\gamma-1}}g_i(t)\\
&+ \frac{\Big(r(t)x+\eta_i(t)x+I_i(t)\Big)e^{-\delta t}a_i(t)}{(x+b_i(t))}-\frac{(\mu(t)-r(t))^2e^{-\delta t}a_i(t)}{2\sigma^2(t)(\gamma-1)}.
\end{aligned}
$$

Adding and subtracting the terms $\frac{r(t)a_i(t)b_i(t)e^{-\delta t}}{x+b_i(t)}$ and $\frac{\eta_i(t)a_i(t)b_i(t)e^{-\delta t}}{x+b_i(t)}$ to the above equation, to obtain

$$
\begin{aligned}
0 &= \frac{e^{-\delta t}a_i(t)}{(x+b_i(t))}\frac{db_i(t)}{dt}+\frac{e^{-\delta t}}{\gamma}\Bigg(\frac{da_i(t)}{dt}-\delta a_i(t)\Bigg)-\lambda_i(t)e^{-\delta t}\frac{a_i(t)}{\gamma}\\
&+ \frac{r(t)xa_i(t)e^{-\delta t}}{(x+b_i(t))}+\frac{r(t)a_i(t)b_i(t)e^{-\delta t}}{x+b_i(t)}-\frac{r(t)a_i(t)b_i(t)e^{-\delta t}}{x+b_i(t)}+\frac{I_i(t)e^{-\delta t}a_i(t)}{x+b_i(t)}\\
&+ \frac{\eta_i(t)xa_i(t)e^{-\delta t}}{(x+b_i(t))}+\frac{\eta_i(t)a_i(t)b_i(t)e^{-\delta t}}{x+b_i(t)}-\frac{\eta_i(t)a_i(t)b_i(t)e^{-\delta t}}{x+b_i(t)}\\
&+ \left(\frac{1-\gamma}{\gamma}\right)e^{-\delta t}\Big(a_i(t)\Big)^{\frac{\gamma}{\gamma-1}}g_i(t)-\frac{(\mu(t)-r(t))^2e^{-\delta t}a_i(t)}{2\sigma^2(t)(\gamma-1)}.
\end{aligned}
$$

Note that the above differential equation can be separated into two independent boundary value problems for a_i and b_i as follows

$$\begin{aligned}
0 &= \frac{e^{-\delta t}}{\gamma}\left(\frac{da_i(t)}{dt} - \delta a_i(t)\right) - \lambda_i(t)e^{-\delta t}\frac{a_i(t)}{\gamma} \\
&+ \frac{r(t)a_i(t)e^{-\delta t}(x+b_i(t))}{x+b_i(t)} + \frac{\eta_i(t)a_i(t)e^{-\delta t}(x+b_i(t))}{x+b_i(t)} \\
&+ \left(\frac{1-\gamma}{\gamma}\right)e^{-\delta t}\Big(a_i(t)\Big)^{\frac{\gamma}{\gamma-1}}g_i(t) - \frac{(\mu(t)-r(t))^2e^{-\delta t}a_i(t)}{2\sigma^2(t)(\gamma-1)},
\end{aligned} \tag{4.33}$$

and

$$\frac{e^{-\delta t}a_i(t)}{(x+b_i(t))}\frac{db_i(t)}{dt} + \frac{I_i(t)e^{-\delta t}a_i(t)}{x+b_i(t)} - \frac{r(t)a_i(t)b_i(t)e^{-\delta t}}{x+b_i(t)} - \frac{\eta_i(t)a_i(t)b_i(t)e^{-\delta t}}{x+b_i(t)} = 0. \tag{4.34}$$

Divide the equation (4.33) by $e^{-\delta t}$ and rearrange the terms, we obtain

$$\begin{aligned}
0 &= \frac{1}{\gamma}\frac{da_i(t)}{dt} + \left(\frac{1-\gamma}{\gamma}\right)\Big(a_i(t)\Big)^{\frac{\gamma}{\gamma-1}}g_i(t) \\
&+ \left(-\frac{\delta}{\gamma} - \frac{\lambda_i(t)}{\gamma} + r(t) + \eta_i(t) - \frac{(\mu(t)-r(t))^2}{2\sigma^2(t)(\gamma-1)}\right)a_i(t), \\
a_i(T) &= w_4.
\end{aligned} \tag{4.35}$$

Multiply equation (4.34) by $\frac{x+b_i(t)}{a_i(t)e^{-\delta t}}$ to get

$$\begin{aligned}
\frac{db_i(t)}{dt} + I_i(t) + \Big(-r(t) - \eta_i(t)\Big)b_i(t) &= 0, \\
b_i(T) &= 0.
\end{aligned} \tag{4.36}$$

To solve equation (4.35) we assume its solution has the form

$$a_i(t) = (l_i(t))^{1-\gamma}. \tag{4.37}$$

Differentiate $a_i(t)$ in equation (4.37) with respect to time t, we get

$$\frac{da_i(t)}{dt} = (1-\gamma)\Big(l_i(t)\Big)^{-\gamma}\frac{dl_i(t)}{dt}. \tag{4.38}$$

Substitute equation (4.38) in equation (4.35) to obtain

$$\begin{aligned}
0 &= \frac{1}{\gamma}(1-\gamma)\Big(l_i(t)\Big)^{-\gamma}\frac{dl_i(t)}{dt} + \left(\frac{1-\gamma}{\gamma}\right)(l_i(t))^{-\gamma}g_i(t) \\
&+ \left(-\frac{\delta}{\gamma} - \frac{\lambda_i(t)}{\gamma} + r(t) + \eta_i(t) - \frac{(\mu(t)-r(t))^2}{2\sigma^2(t)(\gamma-1)}\right)(l_i(t))^{1-\gamma}.
\end{aligned}$$

The above equation can be rewritten as

$$\begin{aligned} \frac{dl_i(t)}{dt} - h_i(t)l_i(t) + g_i(t) &= 0, \\ l_i(T) &= w_4^{\frac{1}{1-\gamma}}, \end{aligned} \tag{4.39}$$

where

$$h_i(t) = \frac{1}{1-\gamma}\Bigg(\delta + \lambda_i(t) - \gamma\left(r(t) + \eta_i(t)\right) - \frac{\gamma}{2}\frac{(\mu(t) - r(t))^2}{\sigma^2(t)(1-\gamma)}\Bigg).$$

Equation (4.39) is linear and 1^{st} order ODE, then we can solve it explicitly by using the integrating factor method to get

$$l_i(t) = e^{\int_T^t h_i(s)ds}\bigg(\int_T^t -g_i(s)e^{-\int_T^s h_i(z)dz}ds + c_1\bigg).$$

Using the condition $l_i(T) = w_4^{\frac{1}{1-\gamma}}$ we obtain

$$l_i(t) = w_4^{\frac{1}{1-\gamma}}e^{-\int_t^T h_i(s)ds} + \int_t^T e^{-\int_t^s h_i(z)\mathrm{d}z}g_i(s)\mathrm{d}s.$$

Substitute the value of $l_i(t)$ in equation (4.37) we get

$$a_i(t) = \bigg(w_4^{\frac{1}{1-\gamma}}e^{-\int_t^T h_i(s)ds} + \int_t^T e^{-\int_t^s h_i(z)\mathrm{d}z}g_i(s)\mathrm{d}s\bigg)^{1-\gamma}.$$

To find a solution for the boundary value problem (4.36), it is again 1^{st} order linear ODE and it can be solved also using the integrating factor method as

$$b_i(t) = e^{-\int_T^t(-r(z)-\eta_i(z))dz}\bigg(-\int_T^t I_i(s)e^{\int_T^s(-r(z)-\eta_i(z))dz}ds + c_2\bigg),$$

Since $b_i(T) = 0$, then $c_2 = 0$. Consequently

$$b_i(t) = \int_t^T I_i(s)e^{-\int_t^s(r(z)+\eta_i(z))dz}\mathrm{d}s.$$

From equation (4.21), we know that

$$c_i^*(t,x) = \left(\frac{V_{i,x}(t,x)e^{\delta t}}{w_i}\right)^{\frac{1}{\gamma-1}}.$$

Substitute the value of function $V_{i,x}(t,x)$ to get

$$\begin{aligned} c_i^*(t,x) &= \left(\frac{e^{-\delta t} a_i(t) \left(x + b_i(t)\right)^{\gamma - 1} e^{\delta t}}{w_i} \right)^{\frac{1}{\gamma-1}} \\ &= \left(\frac{a_i(t) \left(x + b_i(t)\right)^{\gamma - 1}}{w_i} \right)^{\frac{1}{\gamma-1}}. \end{aligned}$$

Substitute the value of $a_i(t)$ from equation (4.37) to get

$$c_i^*(t,x) = w_i^{-\frac{1}{\gamma-1}} \frac{1}{l_i(t)} \left(x + b_i(t)\right).$$

Also from equation (4.26), we know this

$$u_i^*(t,x) = -\frac{(\mu(t) - r(t)) V_{i,x}(t,x)}{\sigma^2(t) V_{i,xx}(t,x)}.$$

Substituting the value of $V_{i,x}(t,x)$ and $V_{i,xx}(t;x)$ from (4.32) in the above equation we get

$$u_i^*(t,x) = -\frac{(\mu(t) - r(t)) e^{-\delta t} a_i(t) \left(x + b_i(t)\right)^{\gamma-1}}{\sigma^2(t)(\gamma-1) e^{-\delta t} a_i(t) \left(x + b_i(t)\right)^{\gamma-2}}.$$

That is,

$$u_i^*(t,x) = -\frac{\mu(t) - r(t)}{\sigma^2(t)(\gamma-1)} \left(x + b_i(t)\right).$$

Similarly, substitute the value of $V_{i,x}(t)$ from (4.32) in equation (4.23), to get

$$k_i^*(t,x) = \left(\left(\frac{\eta_i(t)}{\lambda_i(t) w_3} e^{-\delta t} a_i(t) \left(x + b_i(t)\right)^{\gamma-1} e^{\delta t} \right)^{\frac{1}{\gamma-1}} - x \right) \eta_i(t).$$

Thus,

$$k_i^*(t,x) = \eta_i(t) \Bigg(\left(\frac{\eta_i(t)}{\lambda_i(t) w_3} \right)^{\frac{1}{\gamma-1}} \frac{1}{l_i(t)} \left(x + b_i(t)\right) - x \Bigg).$$

Finally, substitute the value of $V_{i,x}(t)$ from (4.32) in equation (4.25), to get

$$q_i^*(t,x) = \left(\frac{h_i(t)}{\lambda_i(t) w_3} e^{-\delta t} a_i(t) \left(x + b_i(t)\right)^{\gamma-1} e^{\delta t} \right)^{\frac{1}{\gamma-1}} h_i(t).$$

That is,

$$q_i^*(t,x) = h_i(t) \left(\frac{h_i(t)}{\lambda_i(t) w_3} \right)^{\frac{1}{\gamma-1}} \frac{1}{l_i(t)} \left(x + b_i(t)\right),$$

which completes the proof. □

Chapter 5

Optimization problem before the first death

In this chapter, we will consider the optimization problem before first death when the two wage earners are contributing in the social security system while participating in the life-insurance markets. We will use stochastic mortality models for dependent lives, mainly Copula model, to handle a stochastic optimal control problem under consideration.

To study the optimization problem where the two wage earners are alive and to distinguish the strategies here from the strategies of the problem after first death as in Chapter 4, we will use another notations $\bar{c}_i(\cdot), \bar{u}_i(\cdot), \bar{k}_i(\cdot)$ and $\bar{q}_i(\cdot)$, for the representation of consumption, amount invested in the risky asset, life-insurance and welfare, respectively.

5.1 Stochastic optimal control problem before the first death

In this section, we will consider the optimal control problem for the two wage earners in which neither of them dies before the retirement date and we find the optimal strategies that maximize the expected utility for each agent, where both individuals are wage-earners and contribute in the social security system with their partner nominated as the beneficiary.

The wealth process for two wage-earners before first death can be expressed as

$$
\begin{aligned}
X(t) =& x_0 + \int_0^t \left(r(s)X(s) + (\mu(s) - r(s))\bar{u}(s) + \sum_{i=1}^{2} \left(-\bar{c}_i(s) - \bar{k}_i(s) - \bar{q}_i(s) + I_i(s)\right) \right) \mathrm{d}s \\
&+ \int_0^t \sigma(s)\bar{u}(s)\mathrm{d}W(s),
\end{aligned}
\tag{5.1}
$$

where the wealth $X(t)$ is as given in (3.11).

Differentiate equation (5.1) with respect to t we get the following differential form

$$
\begin{aligned}
\mathrm{d}X(t) = &\left(r(t)X(t) + (\mu(t) - r(t))\,\bar{u}(t) + \sum_{i=1}^{2} \left(-\bar{c}_i(t) - \bar{k}_i(t) - \bar{q}_i(t) + I_i(t)\right) \right) \mathrm{d}t \\
&+ \sigma(t)\bar{u}(t)\mathrm{d}W(t).
\end{aligned}
$$

5.2 Tower rule of conditional expectations

In this section, we will introduce some conditional densities of probability distributions functions for our wage-earner. This will help us to formulate the optimal control under consideration and state the desire results before first death.

Based on equations (3.5), (3.6), (3.7) and (3.8) the following lemma holds.

Lemma 5.1. *For any $s, s_1, s_2 \geq t$ the conditional density functions can be written as*

$$
\begin{aligned}
f_1(s;t) &= \frac{f_1(s)}{1 - F_1(t)}, \\
f_2(s;t) &= \frac{f_2(s)}{1 - F_2(t)}, \\
f_{T_1}(s;t) &= \frac{f_{T_1}(s)}{1 - F_{T_1}(t)}, && (5.2) \\
f\left(s_1, s_2; t\right) &= \frac{f\left(s_1, s_2\right)}{1 - F_{T_1}(t)}. && (5.3)
\end{aligned}
$$

Proof. Recall $F_1(s;t)$ is the conditional probability for the first wage-earner time of death to occur at time s conditional upon being alive at time $t \leq s$. From (3.5) we have

$$
\begin{aligned}
F_1(s;t) &= P(\tau_1 \leq s | \tau_1 > t) \\
&= \frac{P(\tau_1 \leq s)}{P(\tau_1 > t)} \\
&= \frac{F_1(s)}{1 - F_1(t)}.
\end{aligned}
\tag{5.4}
$$

Since $f_1(s;t)$ is the density function corresponds to the distribution function $F_1(s;t)$, it follows that

$$f_1(s;t) = \frac{d}{ds} F_1(s;t). \tag{5.5}$$

After substitute the value of $F_1(s;t)$ from equation (5.4) in equation (5.5) we get

$$\begin{aligned} f_1(s;t) &= \frac{d}{ds}\left(\frac{F_1(s)}{1-F_1(t)}\right) \\ &= \frac{1}{1-F_1(t)}\frac{d}{ds}\left(F_1(s)\right). \\ &= \frac{f_1(s)}{1-F_1(t)}, \end{aligned}$$

where $f_1(s)$ is the density function corresponds to the distribution function $F_1(s)$.

Similarly, we derive the densities $f_2(s;t)$, $f_{T_1}(s;t)$, $f(s_1,s_2;t)$. □

Proposition 5.1. *Assume the joint probability distribution function of death times τ_1 and τ_2 is as defined in Definition 2.62 by*

$$F(t,t) = C(F_1(t), F_2(t)),$$

where $C(\cdot,\cdot)$ is Copula function, then the distribution function for the time T_1 is given by

$$F_{T_1}(t) = F_1(t) + F_2(t) - F(t,t).$$

Proof. From Copula Definition 2.59 we have

$$\begin{aligned} F(t,t) &= C(F_1(t), F_2(t)) \\ &= P(\tau_1 \le t, \tau_2 \le t) \\ &= P(\tau_1 \le t) + P(\tau_1 \le t) - P(\tau_1 < t, \tau_2 < t). \end{aligned} \tag{5.6}$$

As we know that from equation (3.3) the marginal probability distribution function for τ_i is F_i, and $F_{T_1}(t) = P(\tau_1 < t, \tau_2 < t)$ is the distribution function for T_1, so, equation (5.6) becomes

$$F(t,t) = F_1(t) + F_2(t) - F_{T_1}(t). \tag{5.7}$$

Rearrange equation (5.7) to get

$$F_{T_1}(t) = F_1(t) + F_2(t) - F(t,t).$$

This completes the proof. □

The following lemma helps us to state a DPP.

Lemma 5.2. *Let $F(\cdot,\cdot)$ be the joint distribution function of death times τ_1 and τ_2 with corresponding density function $f(\cdot,\cdot)$. If τ_1 and τ_2 are independent of the natural filtration generated by the Brownian motion $W(\cdot)$, then*

$$
\begin{aligned}
J(t,x;\pi(\cdot)) &= \frac{1}{1-F_{T_1}(t)}\mathbb{E}_t\Bigg[\int_t^T (1-F_{T_1}(s))\, e^{-\delta s}\Big(w_1 U\left(\bar{c}_1(s)\right) + w_2 U\left(\bar{c}_2(s)\right)\Big)\mathrm{d}s \\
&+ \int_t^T \left(\int_z^\infty f(s,z)\mathrm{d}s\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^T \left(\int_s^\infty f(s,z)\mathrm{d}z\right) V_2\left(s, X(s) + \frac{\bar{k}_1(s)}{\eta_1(s)} + \frac{\bar{q}_1(s)}{h_1(s)}\right)\mathrm{d}s \\
&+ (1-F_{T_1}(T))\, w_4 e^{-\delta T} U(X(T))\Bigg].
\end{aligned} \tag{5.8}
$$

Proof. Based on equation (3.14) we defined $J(t,x;\pi)$ as

$$
\begin{aligned}
J(t,x;\pi(\cdot)) = \mathbb{E}_t\Bigg[&\int_t^{\tau_1\wedge T} w_1 e^{-\delta s} U\left(c_1(s)\right)\mathrm{d}s \\
&+ \int_t^{\tau_2\wedge T} w_2 e^{-\delta s} U\left(c_2(s)\right)\mathrm{d}s + w_3 \mathbf{1}_{\{\tau_1\vee\tau_2\le T\}} e^{-\delta(\tau_1\vee\tau_2)} \\
&\times \left(U\left(X\left(\tau_1\vee\tau_2\right) + \sum_{i=1}^{2}\frac{k_i\left(\tau_i\right)}{\eta_i\left(\tau_i\right)}\mathbf{1}_{\{\tau_i=\tau_1\vee\tau_2\}}\right) + U\left(\sum_{i=1}^{2}\frac{q_i\left(\tau_i\right)}{h_i\left(\tau_i\right)}\mathbf{1}_{\{\tau_i=\tau_1\vee\tau_2\}}\right)\right) \\
&+ w_4 \mathbf{1}_{\{\tau_1\vee\tau_2>T\}} e^{-\delta T} U(X(T))\Bigg].
\end{aligned} \tag{5.9}
$$

Start with the first two terms, we can separate the consumption integrals as follows

$$
\begin{aligned}
& \int_t^{\tau_1\wedge T} w_1 e^{-\delta s} U\left(c_1(s)\right)\mathrm{d}s + \int_t^{\tau_2\wedge T} w_2 e^{-\delta s} U\left(c_2(s)\right)\mathrm{d}s \\
= & \int_t^{T_1\wedge T} w_1 e^{-\delta s} U\left(\bar{c}_1(s,X(s))\right)\mathrm{d}s + \int_t^{T_1\wedge T} w_2 e^{-\delta s} U\left(\bar{c}_2(s,X(s))\right)\mathrm{d}s \\
+ & \mathbf{1}_{\{T_1=\tau_2<\tau_1, T_1\le T\}} \int_{T_1}^{\tau_1\wedge T} w_1 e^{-\delta s} U\left(c_1^*(s,X(s))\right)\mathrm{d}s \\
+ & \mathbf{1}_{\{T_1=\tau_1<\tau_2, T_1\le T\}} \int_{T_1}^{\tau_2\wedge T} w_2 e^{-\delta s} U\left(c_2^*(s,X(s))\right)\mathrm{d}s.
\end{aligned} \tag{5.10}
$$

Now, the third term can be written as

$$
\begin{aligned}
&\mathbf{1}_{\{\tau_1\vee\tau_2\le T\}}e^{-\delta(\tau_1\vee\tau_2)}\\
&\times\left(U\left(X\left(\tau_1\vee\tau_2\right)+\textstyle\sum_{i=1}^{2}\frac{k_i(\tau_i)}{\eta_i(\tau_i)}\mathbf{1}_{\{\tau_i=\tau_1\vee\tau_2\}}\right)+U\left(\textstyle\sum_{i=1}^{2}\frac{q_i(\tau_i)}{h_i(\tau_i)}\mathbf{1}_{\{\tau_i=\tau_1\vee\tau_2\}}\right)\right)\\
&=\mathbf{1}_{\{T_1=\tau_2<\tau_1\le T\}}e^{-\delta\tau_1}\left(U\left(X\left(\tau_1\right)+\tfrac{k_1^*(\tau_1,X(\tau_1))}{\eta_1(\tau_1)}\right)+U\left(\tfrac{q_1^*(\tau_1,X(\tau_1))}{h_1(\tau_1)}\right)\right)\\
&+\mathbf{1}_{\{T_1=\tau_1<\tau_2\le T\}}e^{-\delta\tau_2}\left(U\left(X\left(\tau_2\right)+\tfrac{k_2^*(\tau_2,X(\tau_2))}{\eta_2(\tau_2)}\right)+U\left(\tfrac{q_2^*(\tau_2,X(\tau_2))}{h_2(\tau_2)}\right)\right).
\end{aligned}
\tag{5.11}
$$

Finally, from the fourth term we get that

$$
\begin{aligned}
\mathbf{1}_{\{\tau_1\vee\tau_2>T\}}e^{-\delta T}U(X(T)) &= \left(\mathbf{1}_{\{T_1=\tau_2\le T<\tau_1\}}+\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}\right)e^{-\delta T}U(X(T))\\
&+\left(\mathbf{1}_{\{T_1=\tau_1\le T<\tau_2\}}+\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}\right)e^{-\delta T}U(X(T)).
\end{aligned}
\tag{5.12}
$$

Substitute equations (5.10), (5.11) and (5.12) in $J(t,x;\pi)$ from equation (5.9) to obtain

$$
\begin{aligned}
J(t,x;\pi)=\mathbb{E}_t\Bigg[&\int_t^{T_1\wedge T}w_1e^{-\delta s}U\left(\bar{c}_1(s,X(s))\right)\mathrm{d}s+\int_t^{T_1\wedge T}w_2e^{-\delta s}U\left(\bar{c}_2(s,X(s))\right)\mathrm{d}s\\
&+\mathbf{1}_{\{T_1=\tau_2<\tau_1,T_1\le T\}}\int_{T_1}^{\tau_1\wedge T}w_1e^{-\delta s}U\left(c_1^*(s,X(s))\right)\mathrm{d}s\\
&+\mathbf{1}_{\{T_1=\tau_1<\tau_2,T_1\le T\}}\int_{T_1}^{\tau_2\wedge T}w_2e^{-\delta s}U\left(c_2^*(s,X(s))\right)\mathrm{d}s\\
&+w_3\mathbf{1}_{\{T_1=\tau_2<\tau_1\le T\}}e^{-\delta\tau_1}\left(U\left(X\left(\tau_1\right)+\frac{k_1^*\left(\tau_1,X\left(\tau_1\right)\right)}{\eta_1\left(\tau_1\right)}\right)+U\left(\frac{q_1^*\left(\tau_1,X\left(\tau_1\right)\right)}{h_1\left(\tau_1\right)}\right)\right)\\
&+w_3\mathbf{1}_{\{T_1=\tau_1<\tau_2\le T\}}e^{-\delta\tau_2}\left(U\left(X\left(\tau_2\right)+\frac{k_2^*\left(\tau_2,X\left(\tau_2\right)\right)}{\eta_2\left(\tau_2\right)}\right)+U\left(\frac{q_2^*\left(\tau_2,X\left(\tau_2\right)\right)}{h_2\left(\tau_2\right)}\right)\right)\\
&+w_4\left(\mathbf{1}_{\{T_1=\tau_2\le T<\tau_1\}}+\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}\right)e^{-\delta T}U(X(T))\\
&+w_4\left(\mathbf{1}_{\{T_1=\tau_1\le T<\tau_2\}}+\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}\right)e^{-\delta T}U(X(T))\Bigg].
\end{aligned}
$$

Rearrange terms in the last identity we get

$$
\begin{aligned}
J(t,x;\pi) =&\mathbb{E}_t\Bigg[\int_t^{T_1\wedge T} w_1 e^{-\delta s}U\left(\bar{c}_1(s,X(s))\right)\mathrm{d}s + \int_t^{T_1\wedge T} w_2 e^{-\delta s}U\left(\bar{c}_2(s,X(s))\right)\mathrm{d}s \\
&+\mathbf{1}_{\{T_1=\tau_2<\tau_1,T_1\le T\}}\Bigg(\int_{T_1}^{\tau_1\wedge T} w_1 e^{-\delta s}U\left(c_1^*(s,X(s))\right)\mathrm{d}s \\
&+w_3\mathbf{1}_{\{\tau_1\le T\}}e^{-\delta\tau_1}\left(U\left(X(\tau_1)+\frac{k_1^*(\tau_1,X(\tau_1))}{\eta_1(\tau_1)}\right)+U\left(\frac{q_1^*(\tau_1,X(\tau_1))}{h_1(\tau_1)}\right)\right) \\
&+w_4\mathbf{1}_{\{\tau_1>T\}}e^{-\delta T}U(X(T))\Bigg) \\
&+\mathbf{1}_{\{T_1=\tau_1<\tau_2,T_1\le T\}}\Bigg(\int_{T_1}^{\tau_2\wedge T} w_2 e^{-\delta s}U\left(c_2^*(s,X(s))\right)\mathrm{d}s \\
&+w_3\mathbf{1}_{\{\tau_2\le T\}}e^{-\delta\tau_2}\left(U\left(X(\tau_2)+\frac{k_2^*(\tau_2,X(\tau_2))}{\eta_2(\tau_2)}\right)+U\left(\frac{q_2^*(\tau_2,X(\tau_2))}{h_2(\tau_2)}\right)\right) \\
&+w_4\mathbf{1}_{\{\tau_2>T\}}e^{-\delta T}U(X(T))\Bigg) \\
&+\left\{\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}+\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}\right\}w_4 e^{-\delta T}U(X(T))\Bigg].
\end{aligned}
$$

Based on equation (4.1) $J(t,x;\pi)$ become

$$
\begin{aligned}
J(t,x;\pi) = \mathbb{E}_t\Bigg[&\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s \\
+\quad &\mathbf{1}_{\{T_1=\tau_2<\tau_1,T_1\le T\}}V_1\left(T_1, X(T_1)+\frac{\bar{k}_2(T_1,X(T_1))}{\eta_2(T_1)}+\frac{\bar{q}_2(T_1,X(T_1))}{h_2(T_1)}\right) \\
+\quad &\mathbf{1}_{\{T_1=\tau_1<\tau_2,T_1\le T\}}V_2\left(T_1, X(T_1)+\frac{\bar{k}_1(T_1,X(T_1))}{\eta_1(T_1)}+\frac{\bar{q}_1(T_1,X(T_1))}{h_1(T_1)}\right) \\
+\quad &\left\{\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}+\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}\right\}w_4 e^{-\delta T}U(X(T))\Bigg].
\end{aligned}
\tag{5.13}
$$

The first part of equation (5.13) can be written as

$$
\begin{aligned}
&\mathbb{E}_t\left[\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\right] \\
&=\mathbb{E}_t\Bigg[\mathbf{1}_{\{t<T_1\le T\}}\int_t^{T_1} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s \\
&+\mathbf{1}_{\{T_1>T\}}\int_t^{T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\Bigg].
\end{aligned}
$$

Using the conditional probability function $F_{T_1}(s;t)$ from equation (3.7) and the corresponding conditional density function $f_{T_1}(s;t)$, the last expectation becomes

$$
\begin{aligned}
&\mathbb{E}_t\bigg[\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg]\\
&=\mathbb{E}_t\bigg[\int_t^{T} f_{T_1}(z;t)\mathrm{d}z\int_t^{z} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\\
&+\Big(1-\int_t^{T} f_{T_1}(z;t)\mathrm{d}z\Big)\int_t^{T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg].
\end{aligned}
\tag{5.14}
$$

By the Fubini-Tonelli Theorem 2.40, and since

$$
f_{T_1}(z;t)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\geq 0,
$$

the order of integration can be interchanged as

$$
\begin{aligned}
&\int_t^{T} f_{T_1}(z;t)\mathrm{d}z\int_t^{z} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\\
&=\int_t^{T}\int_t^{z} f_{T_1}(z;t)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\mathrm{d}z\\
&=\int_t^{T}\int_s^{T} f_{T_1}(z;t)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}z\mathrm{d}s\\
&=\int_t^{T}\Big(\int_s^{T} f_{T_1}(z;t)\mathrm{d}z\Big)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s.
\end{aligned}
$$

Now substitute the above equation in equation (5.14) we get

$$
\begin{aligned}
&\mathbb{E}_t\bigg[\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg]\\
&=\mathbb{E}_t\bigg[\int_t^{T}\Big(\int_s^{T} f_{T_1}(z;t)\mathrm{d}z\Big)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\\
&+\Big(1-\int_t^{T} f_{T_1}(z;t)\mathrm{d}z\Big)\int_t^{T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg].
\end{aligned}
$$

Using equation (5.2) in Lemma 5.1, we get

$$
\begin{aligned}
&\mathbb{E}_t\bigg[\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg]\\
&=\mathbb{E}_t\bigg[\int_t^{T}\Big(\int_s^{T}\frac{f_{T_1}(z)}{1-F_{T_1}(t)}\mathrm{d}z\Big)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\\
&+\Big(1-\int_t^{T}\frac{f_{T_1}(z)}{1-F_{T_1}(t)}\mathrm{d}z\Big)\int_t^{T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg].
\end{aligned}
$$

Since $1-F_{T_1}(t)$ doesn't depend on z, it follows that

$$\begin{aligned}&\mathbb{E}_t\bigg[\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg]\\&=\mathbb{E}_t\bigg[\int_t^{T}\left(\frac{\int_s^T f_{T_1}(z)\mathrm{d}z}{1-F_{T_1}(t)}\right)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\\&+\left(1-\frac{\int_t^T f_{T_1}(z)\mathrm{d}z}{1-F_{T_1}(t)}\right)\int_t^T e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg].\end{aligned}$$

As we mentioned previously, f_{T_1} is the density function of F_{T_1} so we rewrite the last expectation as

$$\begin{aligned}&\mathbb{E}_t\bigg[\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg]\\&=\mathbb{E}_t\bigg[\int_t^{T}\left(\frac{F_{T_1}(T)-F_{T_1}(s)}{1-F_{T_1}(t)}\right)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\\&+\left(1-\frac{F_{T_1}(T)-F_{T_1}(t)}{1-F_{T_1}(t)}\right)\int_t^T e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg].\end{aligned}$$

Rearrange terms we get

$$\begin{aligned}&\mathbb{E}_t\bigg[\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg]\\&=\mathbb{E}_t\bigg[\int_t^{T}\left(\frac{F_{T_1}(T)-F_{T_1}(s)+1-F_{T_1}(t)-F_{T_1}(T)+F_{T_1}(t)}{1-F_{T_1}(t)}\right)e^{-\delta s}\\&\times\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg].\end{aligned}$$

These computations lead to

$$\begin{aligned}&\mathbb{E}_t\bigg[\int_t^{T_1\wedge T} e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg]\\&=\mathbb{E}_t\bigg[\int_t^{T}\left(\frac{1-F_{T_1}(s)}{1-F_{T_1}(t)}\right)e^{-\delta s}\Big(w_1U\left(\bar{c}_1(s,X(s))\right)+w_2U\left(\bar{c}_2(s,X(s))\right)\Big)\mathrm{d}s\bigg].\end{aligned}\tag{5.15}$$

Now, the second part of equation (5.13) can be written as

$$\begin{aligned}&\mathbb{E}_t\left[\mathbf{1}_{\{T_1=\tau_2<\tau_1,T_1\le T\}}V_1\left(T_1,X\left(T_1\right)+\frac{\bar{k}_2\left(T_1,X\left(T_1\right)\right)}{\eta_2\left(T_1\right)}+\frac{\bar{q}_2\left(T_1,X\left(T_1\right)\right)}{h_2\left(T_1\right)}\right)\right]\\&=\mathbb{E}_t\bigg[\left\{\mathbf{1}_{\{\tau_2\le T,\tau_1>T\}}+\mathbf{1}_{\{\tau_2\le\tau_1\le T\}}\right\}V_1\left(T_1,X\left(T_1\right)+\frac{\bar{k}_2\left(T_1,X\left(T_1\right)\right)}{\eta_2\left(T_1\right)}+\frac{\bar{q}_2\left(T_1,X\left(T_1\right)\right)}{h_2\left(T_1\right)}\right)\bigg].\end{aligned}$$

Based on the conditional probability function $F(s_1, s_2; t)$ from equation (3.8) and the corresponding conditional density function $f(s_1, s_2; t)$ we write

$$\begin{aligned}
&\mathbb{E}_t\left[\mathbf{1}_{\{T_1=\tau_2<\tau_1, T_1\le T\}} V_1\left(T_1, X(T_1) + \frac{\bar{k}_2(T_1, X(T_1))}{\eta_2(T_1)} + \frac{\bar{q}_2(T_1, X(T_1))}{h_2(T_1)}\right)\right] \\
&= \mathbb{E}_t\Bigg[\int_t^T \left(\int_T^\infty f(s,z;t)\mathrm{d}s\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z,X(z))}{\eta_2(z)} + \frac{\bar{q}_2(z,X(z))}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^T \left(\int_z^T f(s,z;t)\mathrm{d}s\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z,X(z))}{\eta_2(z)} + \frac{\bar{q}_2(z,X(z))}{h_2(z)}\right)\mathrm{d}z\Bigg].
\end{aligned}$$

Using equation (5.3) in Lemma 5.1, we get

$$\begin{aligned}
&\mathbb{E}_t\left[\mathbf{1}_{\{T_1=\tau_2<\tau_1, T_1\le T\}} V_1\left(T_1, X(T_1) + \frac{\bar{k}_2(T_1, X(T_1))}{\eta_2(T_1)} + \frac{\bar{q}_2(T_1, X(T_1))}{h_2(T_1)}\right)\right] \\
&= \mathbb{E}_t\Bigg[\int_t^T \left(\int_T^\infty \frac{f(s,z)}{1-F_{T_1}(t)}\mathrm{d}s\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z,X(z))}{\eta_2(z)} + \frac{\bar{q}_2(z,X(z))}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^T \left(\int_z^T \frac{f(s,z)}{1-F_{T_1}(t)}\mathrm{d}s\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z,X(z))}{\eta_2(z)} + \frac{\bar{q}_2(z,X(z))}{h_2(z)}\right)\mathrm{d}z\Bigg].
\end{aligned}$$

Since $1 - F_{T_1}(t)$ doesn't depend on s we obtain

$$\begin{aligned}
&\mathbb{E}_t\left[\mathbf{1}_{\{T_1=\tau_2<\tau_1, T_1\le T\}} V_1\left(T_1, X(T_1) + \frac{\bar{k}_2(T_1, X(T_1))}{\eta_2(T_1)} + \frac{\bar{q}_2(T_1, X(T_1))}{h_2(T_1)}\right)\right] \\
&= \mathbb{E}_t\Bigg[\int_t^T \left(\frac{\int_T^\infty f(s,z)\,\mathrm{d}s}{1-F_{T_1}(t)}\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z,X(z))}{\eta_2(z)} + \frac{\bar{q}_2(z,X(z))}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^T \left(\frac{\int_z^T f(s,z)\,\mathrm{d}s}{1-F_{T_1}(t)}\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z,X(z))}{\eta_2(z)} + \frac{\bar{q}_2(z,X(z))}{h_2(z)}\right)\mathrm{d}z\Bigg].
\end{aligned} \tag{5.16}$$

Combining the integrals as

$$\int_z^\infty f(s,z)\,\mathrm{d}s = \int_z^T f(s,z)\,\mathrm{d}s + \int_T^\infty f(s,z)\,\mathrm{d}s,$$

to rewrite equation (5.16) as

$$\begin{aligned}
&\mathbb{E}_t\left[\mathbf{1}_{\{T_1=\tau_2<\tau_1, T_1\le T\}} V_1\left(T_1, X(T_1) + \frac{\bar{k}_2(T_1, X(T_1))}{\eta_2(T_1)} + \frac{\bar{q}_2(T_1, X(T_1))}{h_2(T_1)}\right)\right] \\
&= \mathbb{E}_t\Bigg[\int_t^T \left(\frac{\int_z^\infty f(s,z)\mathrm{d}s}{1-F_{T_1}(t)}\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z,X(z))}{\eta_2(z)} + \frac{\bar{q}_2(z,X(z))}{h_2(z)}\right)\mathrm{d}z\Bigg].
\end{aligned} \tag{5.17}$$

The third part of equation (5.13) is similar to second part, thus,

$$\begin{aligned}
&\mathbb{E}_t\left[\mathbf{1}_{\{T_1=\tau_1<\tau_2, T_1\le T\}} V_2\left(T_1, X(T_1) + \frac{\bar{k}_1(T_1, X(T_1))}{\eta_1(T_1)} + \frac{\bar{q}_1(T_1, X(T_1))}{h_1(T_1)}\right)\right] \\
&= \mathbb{E}_t\Bigg[\int_t^T \left(\frac{\int_s^\infty f(s,z)\mathrm{d}z}{1-F_{T_1}(t)}\right) V_2\left(s, X(s) + \frac{\bar{k}_1(s,X(s))}{\eta_1(s)} + \frac{\bar{q}_1(s,X(s))}{h_1(s)}\right)\mathrm{d}s\Bigg].
\end{aligned} \tag{5.18}$$

Finally, the fourth part of equation (5.13) can be written as

$$\mathbb{E}_t\left[w_4\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}e^{-\delta T}U(X(T))+w_4\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}e^{-\delta T}U(X(T))\right]$$
$$=\mathrm{E}_t\left[w_4\mathbf{1}_{\{T<\tau_2<\tau_1\}}e^{-\delta T}U(X(T))+w_4\mathbf{1}_{\{T<\tau_1<\tau_2\}}e^{-\delta T}U(X(T))\right].$$

Based on the conditional probability function $F(s_1,s_2;t)$ from equation (3.8) and the corresponding conditional density function $f(s_1,s_2;t)$, it follows that

$$\mathbb{E}_t\left[w_4\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}e^{-\delta T}U(X(T))+w_4\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}e^{-\delta T}U(X(T))\right]$$
$$=\mathbb{E}_t\left[w_4e^{-\delta T}U(X(T))\int_T^\infty\int_z^\infty f(s,z;t)\mathrm{d}s\ \mathrm{d}z\right.$$
$$\left.+\,w_4e^{-\delta T}U(X(T))\int_T^\infty\int_s^\infty f(s,z;t)\mathrm{d}z\ \mathrm{d}s\right].$$

Using equation (5.3) in Lemma 5.1, we get

$$\mathbb{E}_t\left[w_4\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}e^{-\delta T}U(X(T))+w_4\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}e^{-\delta T}U(X(T))\right]$$
$$=\mathbb{E}_t\left[w_4e^{-\delta T}U(X(T))\int_T^\infty\int_z^\infty \frac{f(s,z)}{1-F_{T_1}(t)}\mathrm{d}s\ \mathrm{d}z\right.$$
$$\left.+\,w_4e^{-\delta T}U(X(T))\int_T^\infty\int_s^\infty \frac{f(s,z)}{1-F_{T_1}(t)}\mathrm{d}z\ \mathrm{d}s\right].$$

Since $1-F_{T_1}(t)$ doesn't depend on s nor z then

$$\mathbb{E}_t\left[w_4\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}e^{-\delta T}U(X(T))+w_4\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}e^{-\delta T}U(X(T))\right]$$
$$=\mathbb{E}_t\left[\frac{w_4e^{-\delta T}U(X(T))}{1-F_{T_1}(t)}\int_T^\infty\int_z^\infty f(s,z)\mathrm{d}s\ \mathrm{d}z\right.$$
$$\left.+\,\frac{w_4e^{-\delta T}U(X(T))}{1-F_{T_1}(t)}\int_T^\infty\int_s^\infty f(s,z)\mathrm{d}z\ \mathrm{d}s\right].$$

Hence,

$$\begin{aligned}&\mathbb{E}_t\left[w_4\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}e^{-\delta T}U(X(T))+w_4\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}e^{-\delta T}U(X(T))\right]\\&=\mathbb{E}_t\left[\frac{1-F_{T_1}(T)}{1-F_{T_1}(t)}w_4e^{-\delta T}U(X(T))\right].\end{aligned}\tag{5.19}$$

Substituting equations (5.15), (5.17), (5.18) and (5.19) in equation (5.13) we obtain equation (5.8), and conclude the proof. □

The previous lemma is the transformation of the control problem in equation (3.14) into a one with a fixed planning horizon, using the tower rules of conditional expectations.

5.3 Hamilton–Jacobi–Bellman equation (HJB)

In this section, we will use Lemma 5.2 to derive first a DPP, and after that we will derive the corresponding HJB equation modeling the problem with all controls before first death. To proceed we assume

$$\tilde{J}(t,x;\pi) = (1 - F_{T_1}(t))\, J(t,x;\pi). \tag{5.20}$$

Then

$$\tilde{V}(t,x) = \sup_{\pi} \tilde{J}(t,x;\pi) = (1 - F_{T_1}(t))\, V(t,x), \tag{5.21}$$

where $\tilde{J}(t,x;\pi)$ denotes the conditional expectation of running and terminal reward functions. Based on equations (5.20) and (5.21) we will introduce the next lemma.

Lemma 5.3. *(DPP) For $0 \le t < s < T$, the maximum expected utility $\tilde{V}(t,x)$ satisfies the recursive relation*

$$\begin{aligned}
\tilde{V}(t,x) &= \sup_{\pi\in\mathcal{A}(t,x)} \mathbb{E}_t\Bigg[\tilde{V}(s, X^{\pi}_{t,x}(s)) \\
&+ \int_t^s (1 - F_{T_1}(u))\, e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u\Bigg].
\end{aligned}$$

Proof. For any $\pi \in \mathcal{A}(t,x)$ with the corresponding wealth $X^{\pi}_{t,x}(\cdot)$, Lemma 5.2 provides that

$$\begin{aligned}
J(t,x;\pi(\cdot)) &= \frac{1}{1 - F_{T_1}(t)}\mathbb{E}_t\Bigg[\int_t^T (1 - F_{T_1}(u))\, e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^T \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^T \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u \\
&+ (1 - F_{T_1}(T))\, w_4 e^{-\delta T} U(X(T))\Bigg].
\end{aligned}$$

Substitute the above equation in equation (5.20) to get

$$\begin{aligned}
\tilde{J}(t,x;\pi(\cdot)) &= \mathbb{E}_t\Bigg[\int_t^T (1-F_{T_1}(u))\, e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^T \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^T \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u \\
&+ (1-F_{T_1}(T))\, w_4 e^{-\delta T} U(X(T))\Bigg].
\end{aligned}$$

For any $0 \le t < s < T$, the above equation becomes

$$\begin{aligned}
\tilde{J}(t,x;\pi(\cdot)) &= \mathbb{E}_t\Bigg[\int_s^T (1-F_{T_1}(u))\, e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_s^T \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_s^T \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u \\
&+ \int_t^s (1-F_{T_1}(u))\, e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u \\
&+ (1-F_{T_1}(T))\, w_4 e^{-\delta T} U(X(T))\Bigg].
\end{aligned}$$

Rearrange the above equation we obtain

$$\begin{aligned}
\tilde{J}(t,x;\pi(\cdot)) &= \mathbb{E}_t\Bigg[\Bigg(\int_s^T (1-F_{T_1}(u))\, e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_s^T \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_s^T \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u \\
&+ (1-F_{T_1}(T))\, w_4 e^{-\delta T} U(X^{\pi}_{t,x}(T))\Bigg) \\
&+ \int_t^s (1-F_{T_1}(u))\, e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u\Bigg].
\end{aligned}$$

The last identity simplifies to

$$\begin{aligned}
\tilde{J}(t,x;\pi(\cdot)) &= \mathbb{E}_t\Bigg[\tilde{J}(s,X^{\pi}_{t,x}(s);\pi(\cdot)) \\
&+ \int_t^s (1-F_{T_1}(u))\,e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right)+w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z)+\frac{\bar{k}_2(z)}{\eta_2(z)}+\frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u)+\frac{\bar{k}_1(u)}{\eta_1(u)}+\frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u\Bigg].
\end{aligned}$$

Using equation (5.21) we obtain

$$\begin{aligned}
\tilde{J}(t,x;\pi(\cdot)) &= \mathbb{E}_t\Bigg[\tilde{J}(s,X^{\pi}_{t,x}(s);\pi(\cdot)) \\
&+ \int_t^s (1-F_{T_1}(u))\,e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right)+w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z)+\frac{\bar{k}_2(z)}{\eta_2(z)}+\frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u)+\frac{\bar{k}_1(u)}{\eta_1(u)}+\frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u\Bigg]. \\
&\leq \mathbb{E}_t\Bigg[\tilde{V}(s,X^{\pi}_{t,x}(s))) \\
&+ \int_t^s (1-F_{T_1}(u))\,e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right)+w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z)+\frac{\bar{k}_2(z)}{\eta_2(z)}+\frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
&+ \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u)+\frac{\bar{k}_1(u)}{\eta_1(u)}+\frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u\Bigg].
\end{aligned}$$

Note that since $\pi(\cdot) = \big(\bar{c}_1(\cdot), \bar{c}_2(\cdot), \bar{k}_1(\cdot), \bar{k}_2(\cdot), \bar{q}_1(\cdot), \bar{q}_2(\cdot), \bar{u}(\cdot)\big)$ is arbitrary, it follows that

$$\begin{aligned}
\tilde{V}(t,x)) &\leq \sup_{\pi\in\mathcal{A}(t,x)} \mathbb{E}_t\Bigg[\tilde{V}(s,X^{\pi}_{t,x}(s)) \\
&+ \int_t^s (1-F_{T_1}(u))\,e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right)+w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
&+ \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z)+\frac{\bar{k}_2(z)}{\eta_2(z)}+\frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \qquad (5.22)\\
&+ \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u)+\frac{\bar{k}_1(u)}{\eta_1(u)}+\frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u\Bigg].
\end{aligned}$$

Conversely given that $\pi(\cdot) \in \mathcal{A}(t,x)$, for $\epsilon > 0$ and $\omega \in \Omega$, using property of supremum there exists

$$\mathcal{L} \equiv \left(\bar{c}_{1,\omega,\epsilon}(\cdot), \bar{c}_{2,\omega,\epsilon}(\cdot), \bar{k}_{1,\omega,\epsilon}(\cdot), \bar{k}_{2,\omega,\epsilon}(\cdot), \bar{q}_{1,\omega,\epsilon}(\cdot), \bar{q}_{2,\omega,\epsilon}(\cdot), \bar{u}_{\omega,\epsilon}(\cdot)\right) \in \mathcal{A}\left(s, X^{\pi}_{t,x}(s,\omega)\right),$$

where

$$\tilde{J}\left(s, X^{\pi}_{t,x}(s); \mathcal{L}_{\omega,\epsilon}(\cdot)\right) \geq \tilde{V}\left(s, X^{\pi}_{t,x}(s)\right) - \epsilon.$$

Let

$$\mathcal{L}^*(u) := \begin{cases} \left(\bar{c}_1(\cdot), \bar{c}_2(\cdot), \bar{k}_1(\cdot), \bar{k}_2(\cdot), \bar{q}_1(\cdot), \bar{q}_2(\cdot), \bar{u}(\cdot)\right), & \text{if } u \in [t,s], \\ \mathcal{L}_{\omega,\epsilon}(u), & \text{if } u \in [s,T]. \end{cases}$$

Notice that $X^{\mathcal{L}^*}_{t,x}(T) = X^{\mathcal{L}_{\omega,c}}_{s,X^{\pi}_{t,x}(s)}(T)$ a.s., then

$$\bar{V}(t,x) \geq \tilde{J}\left(t, x; \mathcal{L}^*(\cdot)\right).$$

Thus,

$$\begin{aligned}
\mathbb{E}_t \quad & \Bigg[\Bigg(\int_s^T \left(1 - F_{T_1}(u)\right) e^{-\delta u}\Big(w_1 U\left(\bar{c}_{1,\omega,\epsilon}(u)\right) + w_2 U\left(\bar{c}_{2,\omega,\epsilon}(u)\right)\Big)\mathrm{d}u \\
+ \quad & \int_s^T \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_{2,\omega,\epsilon}(z)}{\eta_2(z)} + \frac{\bar{q}_{2,\omega,\epsilon}(z)}{h_2(z)}\right)\mathrm{d}z \\
+ \quad & \int_s^T \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_{1,\omega,\epsilon}(u)}{\eta_1(u)} + \frac{\bar{q}_{1,\omega,\epsilon}(u)}{h_1(u)}\right)\mathrm{d}u \\
+ \quad & \left(1 - F_{T_1}(T)\right) w_4 e^{-\delta T} U(X^{\mathcal{L}_{\omega,c}}_{s,X^{\pi}_{t,x}(s)}(T))\Bigg) \\
+ \quad & \int_t^s \left(1 - F_{T_1}(u)\right) e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
+ \quad & \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
+ \quad & \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u\Bigg] \\
\geq \quad & \mathbb{E}_t\Bigg[\tilde{V}\left(s, X^{\pi}_{t,x}(s)\right) - \epsilon \\
+ \quad & \int_t^s \left(1 - F_{T_1}(u)\right) e^{-\delta u}\Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right)\Big)\mathrm{d}u \\
+ \quad & \int_t^s \left(\int_z^\infty f(u,z)\mathrm{d}u\right) V_1\left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)}\right)\mathrm{d}z \\
+ \quad & \int_t^s \left(\int_u^\infty f(u,z)\mathrm{d}z\right) V_2\left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)}\right)\mathrm{d}u\Bigg].
\end{aligned}$$

The above inequality holds for any $\pi(\cdot) \in \mathcal{A}(t,x)$ and $\epsilon > 0$, then

$$\begin{aligned}
\tilde{V}(t,x)) \quad \leq \quad & \sup_{\pi \in \mathcal{A}(t,x)} \mathbb{E}_t \Bigg[\tilde{V}(s, X^{\pi}_{t,x}(s)) \\
+ \quad & \int_t^s (1 - F_{T_1}(u))\, e^{-\delta u} \Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right) \Big) \mathrm{d}u \\
+ \quad & \int_t^s \left(\int_z^{\infty} f(u,z) \mathrm{d}u \right) V_1 \left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)} \right) \mathrm{d}z \\
+ \quad & \int_t^s \left(\int_u^{\infty} f(u,z) \mathrm{d}z \right) V_2 \left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)} \right) \mathrm{d}u \Bigg]. \qquad (5.23)
\end{aligned}$$

Finally, we can get DPP from inequalities (5.22) and (5.23). $\square$

Now, we will use the DPP obtained in previous lemma to derived the following HJB equation.

Theorem 5.4. *(HJB-Equation) Suppose that the maximum expected utility $\tilde{V}(t,x) \in C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$. Then $\tilde{V}(t,x)$ must satisfies the HJB equation*

$$\begin{cases} \tilde{V}_t(t,x) + \sup\limits_{(\bar{c}_1, \bar{c}_2, \bar{k}_1 \bar{k}_2, \bar{q}_1 \bar{q}_2, \bar{u})} \mathcal{H}(t,x;\pi(\cdot)) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ \tilde{V}(T,x) = (1 - F_{T_1}(T))\, w_4 e^{-\delta T} U(X(T)), & x \in \mathbb{R}, \end{cases}$$

where the Hamilltonian function $\mathcal{H}$ is given by

$$\begin{aligned}
\mathcal{H}(t,x;\pi(\cdot)) = & \frac{1}{2}\sigma^2(t)\bar{u}^2 \tilde{V}_{xx} + e^{-\delta t}\left(1 - F_{T_1}(t)\right)\left(w_1 U\left(\bar{c}_1\right) + w_2 U\left(\bar{c}_2\right)\right) + \tilde{V}_x(t,x) \\
& \times \big(r(t)x + (\mu(t) - r(t))\bar{u} - \bar{c}_1 - \bar{k}_1 - \bar{q}_1 + I_1(t) \; - \bar{c}_2 - \bar{k}_2 - \bar{q}_2 + I_2(t) \big) \\
& + V_1 \left(t, x + \frac{\bar{k}_2}{\eta_2(t)} + \frac{\bar{q}_2}{h_2(t)} \right) \int_t^{\infty} f(s,t) \mathrm{d}s \\
& + V_2 \left(t, x + \frac{\bar{k}_1}{\eta_1(t)} + \frac{\bar{q}_1}{h_1(t)} \right) \int_t^{\infty} f(t,z) \mathrm{d}z,
\end{aligned}$$

$\tilde{V}_t$ and $\tilde{V}_x$ denote first-order partial derivatives with respect to t and x, respectively, and $\tilde{V}_{xx}$ denotes a second-order derivative with respect to x. Moreover,

$$\pi^*(\cdot) = \left(\bar{c}_1^*(\cdot), \bar{c}_2^*(\cdot), \bar{k}_1^*(\cdot), \bar{k}_2^*(\cdot), \bar{q}_1^*(\cdot), \bar{q}_2^*(\cdot), \bar{u}^*(\cdot) \right) \in \mathcal{A}(t,x),$$

whose wealth X^* is optimal if and if for $s \in [t,T]$ we have

$$\tilde{V}_t\left(s, X^*(s)\right) + \mathcal{H}\left(s, X^*(s); \pi^*\right) = 0.$$

Proof. Apply $s = t + h$ in the DPP from Lemma 5.3. By Itô's formula Theorem 2.56, we can get

$$\tilde{V}(t+h, X(t+h)) = \tilde{V}(t,x) + \int_t^{t+h} \Big(\tilde{V}_t(s, X(s)) + \tilde{V}_x(s, X(s)) \\ \times \left(r(s)X(s) + (\mu(s) - r(s))\,\bar{u}(s) + \sum_{i=1}^{2} \left(-\bar{c}_i(s) - \bar{k}_i(s) - \bar{q}_i(s) + I_i(s)\right) \right) \\ + \frac{1}{2}\tilde{V}_{xx}(s, X(s))\sigma^2(s)\bar{u}^2(s) \Big) ds + \int_t^{t+h} \tilde{V}_x(s, X(s))\sigma(s)\bar{u}(s) dW(s). \tag{5.24}$$

Using Lemma 5.3 we get

$$\begin{aligned}
0 &= \sup_{\pi \in \mathcal{A}(t,x)} \mathbb{E}_t \Big[\tilde{V}(t+h, X(t+h)) - \tilde{V}(t,x) \\
&+ \int_t^{t+h} \left(1 - F_{T_1}(u)\right) e^{-\delta u} \Big(w_1 U\left(\bar{c}_1(u)\right) + w_2 U\left(\bar{c}_2(u)\right) \Big) \mathrm{d}u \\
&+ \int_t^{t+h} \left(\int_z^{\infty} f(u,z) \mathrm{d}u \right) V_1 \left(z, X(z) + \frac{\bar{k}_2(z)}{\eta_2(z)} + \frac{\bar{q}_2(z)}{h_2(z)} \right) \mathrm{d}z \\
&+ \int_t^{t+h} \left(\int_u^{\infty} f(u,z) \mathrm{d}z \right) V_2 \left(u, X(u) + \frac{\bar{k}_1(u)}{\eta_1(u)} + \frac{\bar{q}_1(u)}{h_1(u)} \right) \mathrm{d}u \Big].
\end{aligned}$$

Now substitute equation (5.24) into the above equation and let h goes to zero to obtain

$$\begin{aligned}
0 &= \sup_{\pi \in \mathcal{A}(t,x)} \mathbb{E}_t \Big[\tilde{V}_t(t,x) + \tilde{V}_x(t,x) \\
&\times \left(r(t)X(t) + (\mu(t) - r(t))\,\bar{u}(t) + \sum_{i=1}^{2} \left(-\bar{c}_i(t) - \bar{k}_i(t) - \bar{q}_i(t) + I_i(t)\right) \right) \\
&+ \frac{1}{2}\tilde{V}_{xx}(t, X(t))\sigma^2(t)\bar{u}^2(t) \Big) \\
&+ \left(1 - F_{T_1}(t)\right) e^{-\delta t} \Big(w_1 U\left(\bar{c}_1(t)\right) + w_2 U\left(\bar{c}_2(t)\right) \Big) \\
&+ \left(\int_t^{\infty} f(s,t) \mathrm{d}s \right) V_1 \left(t, X(t) + \frac{\bar{k}_2(t)}{\eta_2(t)} + \frac{\bar{q}_2(t)}{h_2(t)} \right) \\
&+ \left(\int_t^{\infty} f(t,z) \mathrm{d}z \right) V_2 \left(t, X(t) + \frac{\bar{k}_1(t)}{\eta_1(t)} + \frac{\bar{q}_1(t)}{h_1(t)} \right) \Big].
\end{aligned}$$

Since $\tilde{V}_t(t,x)$ doesn't depend on π, the first part of HJB theorem holds. Now, the proof of the second part of the HJB theorem, is similar to proof of equation (4.10) in Theorem 4.4. This completes the proof. □

5.4 Explicit solution

In this section, we will find an explicit solution for the (HJB) equation (5.4) in some possible cases. To solve the HJB equation (5.4) and derive the corresponding optimal strategies, we assume a power utility function

$$U(x) = \frac{x^\gamma}{\gamma}, \tag{5.25}$$

where $\gamma < 1$.

For problem before the first death, we will determine $\tilde{V}(t,x)$ which has the form

$$\tilde{V}(t,x) = \frac{A(t)}{\gamma}(x + B(t))^\gamma. \tag{5.26}$$

Substitute $\tilde{V}(t,x)$ from equation (5.26) in equation (5.21) and solve for $V(t,x)$ we get

$$V(t,x) = \frac{A(t)}{\gamma\,[1 - F_{T_1}(t)]}(x + B(t))^\gamma. \tag{5.27}$$

The previous function is ansatz function, we will use it in the next results in order to derive an explicit solution.

Unfortunately, it is difficult for us to solve the HJB equation from Theorem 5.4 explicitly when life-insurance and welfare parameters are all control variables. Therefore, we will try to find an explicit solution for following cases

- Case 1: No life-insurance contracts.
- Case 2: No Welfare policy contracts.
- Case 3: Life-insurance is not being control variable.
- Case 4: Welfare policy is not being control variable.

5.4.1 CASE 1: No life-insurance contracts

In this section, we assume no life-insurance contracts available, that is, $\bar{k}_1 = \bar{k}_2 = 0$, we can derived an explicit solutions for the other controls.

The following proposition is an explicit solution of the first case.

Proposition 5.2. *Assume $\bar{k}_1 = \bar{k}_2 = 0$, $U(\cdot)$ is as given in equation (5.25). Conditions of Lemma 5.2 hold, and the value function $V(t,x)$ is as given in equation (5.27). Then the optimal strategies are given by*

$$\begin{cases} \bar{c}_i(t,x) = \left(\frac{e^{\delta t}}{(1-F_{T_1}(t))w_i}\right)^{\frac{1}{\gamma-1}} \frac{1}{D(t)}(x+B(t)), i = 1,2, \\ \bar{u}(t,x) = -\frac{\mu(t)-r(t)}{(\gamma-1)\sigma^2(t)}(x+B(t)), \\ \bar{q}_1(t,x) = h_1(t)\left(\left(\frac{e^{\delta t}h_1(t)}{\int_t^\infty f(t,z)dz}\right)^{\frac{1}{\gamma-1}} \times \frac{l_2(t)}{D(t)}(x+B(t)) - x - b_2(t)\right), \\ \bar{q}_2(t,x) = h_2(t)\left(\left(\frac{e^{\delta t}h_2(t)}{\int_t^\infty f(s,t)ds}\right)^{\frac{1}{\gamma-1}} \times \frac{l_1(t)}{D(t)}(x+B(t)) - x - b_1(t)\right), \end{cases}$$

where

$$\begin{aligned} D(t) &= \left((1-F_{T_1}(T))\, w_4 e^{-\delta T}\right)^{\frac{1}{1-\gamma}} e^{-\int_t^T H(s)\mathrm{d}s} + \int_t^T e^{-\int_t^s H(z)\mathrm{d}z} G(s)\mathrm{d}s, \\ B(t) &= \int_t^T e^{-\int_t^s (r(z)+h_1(z)+h_2(z))\mathrm{d}z} \left(I_1(s)+I_2(s)+h_1(s)b_2(s)+h_2(s)b_1(s)\right)\mathrm{d}s, \\ H(t) &= \frac{1}{1-\gamma}\left(\frac{1}{2}\frac{(\mu(t)-r(t))^2\gamma}{(\gamma-1)\sigma^2(t)} - \gamma\left(r(t)+h_1(t)+h_2(t)\right)\right), \\ G(t) &= e^{\frac{\delta t}{\gamma-1}}\left((1-F_{T_1}(t))^{\frac{1}{1-\gamma}}\left(w_1^{\frac{1}{1-\gamma}} + w_2^{\frac{1}{1-\gamma}}\right) + \left(\frac{h_1(t)}{\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}} l_2(t)\int_t^\infty f(t,z)\mathrm{d}z \right. \\ &\quad \left. + \left(\frac{h_2(t)}{\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}} l_1(t)\int_t^\infty f(s,t)\mathrm{d}s\right). \end{aligned}$$

Proof. From *HJB* Theorem 5.4, $\mathcal{H}$ is being maximum at π^*, so

$$\begin{aligned} &\sup_{(\bar{c}_1,\bar{c}_2,\bar{q}_1,\bar{q}_2,\bar{u})\in(\mathbb{R}^+)^5} \mathcal{H}(t,x;\bar{c}_1,\bar{c}_2,\bar{q}_1,\bar{q}_2,\bar{u}) \\ &= \sup_{(\bar{c}_1,\bar{c}_2)\in\mathbb{R}^+} \left\{ e^{-\delta t}(1-F_{T_1}(t))\, w_1 U(\bar{c}_1(t,x)) - \bar{c}_1(t,x)\tilde{V}_x(t,x) \right. \\ &\left. + e^{-\delta t}(1-F_{T_1}(t))\, w_2 U(\bar{c}_2(t,x)) - \bar{c}_2(t,x)\tilde{V}_x(t,x)\right\} \\ &+ \sup_{(\bar{q}_1,\bar{q}_2)\in\mathbb{R}^+}\left\{ V_1\left(t, x+\frac{\bar{q}_2(t)}{h_2(t)}\right)\int_t^\infty f(s,t)\mathrm{d}s - \bar{q}_2(t)\tilde{V}_x(t,x) \right. \\ &\left. + V_2\left(t, x+\frac{\bar{q}_1(t)}{h_1(t)}\right)\int_t^\infty f(t,z)\mathrm{d}z - \bar{q}_1(t)\tilde{V}_x(t,x)\right\} \\ &+ \sup_{\bar{u}\in\mathbb{R}^+}\left\{\frac{1}{2}\sigma^2(t)\bar{u}^2(t,x)\tilde{V}_{xx}(t,x) + (\mu(t)-r(t))\bar{u}(t,x)\tilde{V}_x(t,x)\right\} \\ &+ \left(r(t)x + I_1(t) + I_2(t)\right)\tilde{V}_x(t,x). \end{aligned} \tag{5.28}$$

Substitute the value of power utility function (5.25) in equation (5.28), and derive $\mathcal{H}$ with respect the variables $\bar{c}_1^*$, $\bar{c}_2^*$ and $\bar{u}^*$, respectively, we obtain that

$$e^{-\delta t}\left(1-F_{T_1}(t)\right) w_i \bar{c}_i^*(t,x)^{\gamma-1} - \tilde{V}_x(t,x) = 0, \quad i=1,2,$$
$$\sigma^2(t)\bar{u}^*(t,x)\tilde{V}_{xx}(t,x) + (\mu(t)-r(t))\tilde{V}_x(t,x) = 0.$$

Rearrange the above equations for $\bar{c}_i^*, i=1,2,$ and $\bar{u}^*$, respectively, we get

$$\begin{aligned} \bar{c}_i^*(t,x) &= \left(\frac{e^{\delta t}\tilde{V}_x(t,x)}{\left(1-F_{T_1}(t)\right) w_i}\right)^{\frac{1}{\gamma-1}}, \quad i=1,2, \\ \bar{u}^*(t,x) &= -\frac{(\mu(t)-r(t))\tilde{V}_x(t,x)}{\sigma^2(t)\tilde{V}_{xx}(t,x)}. \end{aligned} \tag{5.29}$$

Now, to obtain an optimal welfare purchase strategies, we can considered the bivariate function ϕ for $\bar{q}_i(t,x), i=1,2$

$$\begin{aligned} \phi\left(\bar{q}_1,\bar{q}_2\right) = &-\left(\bar{q}_1+\bar{q}_2\right)\tilde{V}_x(t,x) \\ &+ V_1\left(t, x+\frac{\bar{q}_2}{h_2(t)}\right) \times \int_t^\infty f(s,t)\mathrm{d}s \\ &+ V_2\left(t, x+\frac{\bar{q}_1}{h_1(t)}\right) \times \int_t^\infty f(t,z)\mathrm{d}z. \end{aligned}$$

Now, apply the value function defined in the Theorem 4.6 to get

$$\begin{aligned} \phi\left(\bar{q}_1,\bar{q}_2\right) &= -\left(\bar{q}_1+\bar{q}_2\right)\tilde{V}_x(t,x) \\ &+ \frac{1}{\gamma}e^{-\delta t}l_1^{1-\gamma}(t)\left(x+\frac{\bar{q}_2}{h_2(t)}+b_1(t)\right)^\gamma \times \int_t^\infty f(s,t)\mathrm{d}s \\ &+ \frac{1}{\gamma}e^{-\delta t}l_2^{1-\gamma}(t)\left(x+\frac{\bar{q}_1}{h_1(t)}+b_2(t)\right)^\gamma \times \int_t^\infty f(t,z)\mathrm{d}z. \end{aligned} \tag{5.30}$$

Let us compute the first partial derivatives of $\phi\left(\bar{q}_1,\bar{q}_2\right)$ from equation (5.30) to get

$$\begin{aligned} \phi_{\bar{q}_1}\left(\bar{q}_1,\bar{q}_2\right) &= -\tilde{V}_x(t,x) + \frac{e^{-\delta t}}{h_1(t)}l_2^{1-\gamma}(t)\left(x+\frac{\bar{q}_1}{h_1(t)}+b_2(t)\right)^{\gamma-1} \times \int_t^\infty f(t,z)\mathrm{d}z, \\ \phi_{\bar{q}_2}\left(\bar{q}_1,\bar{q}_2\right) &= -\tilde{V}_x(t,x) + \frac{e^{-\delta t}}{h_2(t)}l_1^{1-\gamma}(t)\left(x+\frac{\bar{q}_2}{h_2(t)}+b_1(t)\right)^{\gamma-1} \times \int_t^\infty f(s,t)\mathrm{d}s. \end{aligned}$$

From the above first partial derivatives, we can compute the follows second partial derivatives

$$\begin{aligned}
\phi_{\bar{q}_1\bar{q}_1}\left(\bar{q}_1,\bar{q}_2\right) &= \frac{\gamma-1}{(h_1(t))^2}e^{-\delta t}l_2^{1-\gamma}(t)\left(x+\frac{\bar{q}_1}{h_1(t)}+b_2(t)\right)^{\gamma-2}\times\int_t^{\infty}f(t,z)\mathrm{d}z,\\
\phi_{\bar{q}_2\bar{q}_2}\left(\bar{q}_1,\bar{q}_2\right) &= \frac{\gamma-1}{(h_2(t))^2}e^{-\delta t}l_1^{1-\gamma}(t)\left(x+\frac{\bar{q}_2}{h_2(t)}+b_1(t)\right)^{\gamma-2}\times\int_t^{\infty}f(s,t)\mathrm{d}s, \quad (5.31)\\
\phi_{\bar{q}_1\bar{q}_2}\left(\bar{q}_1,\bar{q}_2\right) &= 0.
\end{aligned}$$

We mentioned previously the $\gamma<1$, that is

$$\gamma<1. \tag{5.32}$$

From (5.31) and (5.32), we can conclude that

$$\phi_{\bar{q}_1\bar{q}_1}\left(\bar{q}_1,\bar{q}_2\right)<0, \tag{5.33}$$

and

$$\phi_{\bar{q}_2\bar{q}_2}\left(\bar{q}_1,\bar{q}_2\right)<0. \tag{5.34}$$

Thus,

$$\phi_{\bar{q}_1\bar{q}_1}\left(\bar{q}_1,\bar{q}_2\right)\phi_{\bar{q}_2\bar{q}_2}\left(\bar{q}_1,\bar{q}_2\right)-\left(\phi_{\bar{q}_1\bar{q}_2}\left(\bar{q}_1,\bar{q}_2\right)\right)^2>0. \tag{5.35}$$

Thus, based on Theorem 2.41, from (5.33),(5.34) and (5.35), then $\phi\left(\bar{q}_1,\bar{q}_2\right)$ admits its maximum at $(\bar{q}_1^*,\bar{q}_2^*)$, which solves the equation

$$\phi_{\bar{q}_1}\left(\bar{q}_1^*,\bar{q}_2^*\right)=\phi_{\bar{q}_2}\left(\bar{q}_1^*,\bar{q}_2^*\right)=0. \tag{5.36}$$

Based on identity (5.36) we see that

$$\begin{aligned}
\left(x+\frac{\bar{q}_1^*}{h_1(t)}+b_2(t)\right)^{\gamma-1} &= \frac{e^{\delta t}\tilde{V}_x(t,x)h_1(t)}{l_2^{1-\gamma}(t)\int_t^{\infty}f(t,z)\mathrm{d}z},\\
\left(x+\frac{\bar{q}_2^*}{h_2(t)}+b_1(t)\right)^{\gamma-1} &= \frac{e^{\delta t}\tilde{V}_x(t,x)h_2(t)}{l_1^{1-\gamma}(t)\int_t^{\infty}f(s,t)\mathrm{d}s}.
\end{aligned} \tag{5.37}$$

Rearranging the above equation, we get

$$\begin{aligned}
\frac{\bar{q}_1^*}{h_1(t)} &= -\left(x+b_2(t)\right)+\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_1(t)}{l_2^{1-\gamma}(t)\int_t^{\infty}f(t,z)\mathrm{d}z}\right)^{\frac{1}{\gamma-1}},\\
\frac{\bar{q}_2^*}{h_2(t)} &= -\left(x+b_1(t)\right)+\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_2(t)}{l_1^{1-\gamma}(t)\int_t^{\infty}f(s,t)\mathrm{d}s}\right)^{\frac{1}{\gamma-1}}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
\bar{q}_1^* &= -h_1(t)\left(x+b_2(t)\right)+h_1(t)\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_1(t)}{l_2^{1-\gamma}(t)\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{1}{\gamma-1}},\\
\bar{q}_2^* &= -h_2(t)\left(x+b_1(t)\right)+h_2(t)\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_2(t)}{l_1^{1-\gamma}(t)\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{1}{\gamma-1}}.
\end{aligned}
\tag{5.38}
$$

Now, substitute (5.37) in (5.30) to get

$$
\begin{aligned}
\phi\left(\bar{q}_1^*,\bar{q}_2^*\right) = &-\left(\bar{q}_1^*+\bar{q}_2^*\right)\tilde{V}_x(t,x)\\
&+\frac{1}{\gamma}e^{-\delta t}l_1^{1-\gamma}(t)\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_2(t)}{l_1^{1-\gamma}(t)\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}}\times\int_t^\infty f(s,t)\mathrm{d}s\\
&+\frac{1}{\gamma}e^{-\delta t}l_2^{1-\gamma}(t)\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_1(t)}{l_2^{1-\gamma}(t)\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}}\times\int_t^\infty f(t,z)\mathrm{d}z.
\end{aligned}
$$

Substitute (5.38) in above equation to get

$$
\begin{aligned}
\phi\left(\bar{q}_1^*,\bar{q}_2^*\right) = &\,h_1(t)\left(x+b_2(t)\right)\tilde{V}_x(t,x)-h_1(t)\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_1(t)}{l_2^{1-\gamma}(t)\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{1}{\gamma-1}}\tilde{V}_x(t,x)\\
&+h_2(t)\left(x+b_1(t)\right)\tilde{V}_x(t,x)-h_2(t)\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_2(t)}{l_1^{1-\gamma}(t)\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{1}{\gamma-1}}\tilde{V}_x(t,x)\\
&+\frac{1}{\gamma}e^{-\delta t}l_1^{1-\gamma}(t)\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_2(t)}{l_1^{1-\gamma}(t)\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}}\times\int_t^\infty f(s,t)\mathrm{d}s\\
&+\frac{1}{\gamma}e^{-\delta t}l_2^{1-\gamma}(t)\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_1(t)}{l_2^{1-\gamma}(t)\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}}\times\int_t^\infty f(t,z)\mathrm{d}z.
\end{aligned}
$$

Rearrange the above terms to obtain

$$
\begin{aligned}
\phi\left(\bar{q}_1^*,\bar{q}_2^*\right) &= \left(h_1\left(x+b_2(t)\right)+h_2\left(x+b_1(t)\right)\right)\tilde{V}_x(t,x)\\
&- h_1\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_1(t)}{l_2^{1-\gamma}(t)\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{1}{\gamma-1}}\tilde{V}_x(t,x)\\
&- h_2\left(\frac{e^{\delta t}\tilde{V}_x(t,x)h_2(t)}{l_1^{1-\gamma}(t)\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{1}{\gamma-1}}\tilde{V}_x(t,x)\\
&+ e^{\frac{\delta t}{\gamma-1}}\left(\tilde{V}_x(t,x)h_2(t)\right)^{\frac{\gamma}{\gamma-1}}\left(\int_t^\infty f(s,t)\mathrm{d}s\right)^{\frac{1}{1-\gamma}}\frac{l_1(t)}{\gamma}\\
&+ e^{\frac{\delta t}{\gamma-1}}\left(\tilde{V}_x(t,x)h_1(t)\right)^{\frac{\gamma}{\gamma-1}}\left(\int_t^\infty f(t,z)\mathrm{d}z\right)^{\frac{1}{1-\gamma}}\frac{l_2(t)}{\gamma}.
\end{aligned}
$$

Simplify the above equation to get

$$\begin{aligned}
\phi\left(\bar{q}_1^*, \bar{q}_2^*\right) &= \left(h_1\left(x+b_2(t)\right)+h_2\left(x+b_1(t)\right)\right)\tilde{V}_x(t,x) \\
&- e^{\frac{\delta t}{\gamma-1}}\left(\tilde{V}_x(t,x)h_1(t)\right)^{\frac{\gamma}{\gamma-1}}\left(\int_t^\infty f(t,z)\mathrm{d}z\right)^{\frac{1}{1-\gamma}} l_2(t) \\
&- e^{\frac{\delta t}{\gamma-1}}\left(\tilde{V}_x(t,x)h_2(t)\right)^{\frac{\gamma}{\gamma-1}}\left(\int_t^\infty f(s,t)\mathrm{d}s\right)^{\frac{1}{1-\gamma}} l_1(t) \\
&+ e^{\frac{\delta t}{\gamma-1}}\left(\tilde{V}_x(t,x)h_2(t)\right)^{\frac{\gamma}{\gamma-1}}\left(\int_t^\infty f(s,t)\mathrm{d}s\right)^{\frac{1}{1-\gamma}} \frac{l_1(t)}{\gamma} \\
&+ e^{\frac{\delta t}{\gamma-1}}\left(\tilde{V}_x(t,x)h_1(t)\right)^{\frac{\gamma}{\gamma-1}}\left(\int_t^\infty f(t,z)\mathrm{d}z\right)^{\frac{1}{1-\gamma}} \frac{l_2(t)}{\gamma}.
\end{aligned}$$

Rearrange the above terms we get

$$\begin{aligned}
\phi\left(\bar{q}_1^*, \bar{q}_2^*\right) &= \left(h_1\left(x+b_2(t)\right)+h_2\left(x+b_1(t)\right)\right)\tilde{V}_x(t,x) - e^{\frac{\delta t}{\gamma-1}}\left(1-\frac{1}{\gamma}\right)\tilde{V}_x^{\frac{\gamma}{\gamma-1}}(t,x) \\
&\times \left(\left(\frac{h_1(t)}{\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}} l_2(t)\int_t^\infty f(t,z)\mathrm{d}z\right. \\
&+ \left.\left(\frac{h_2(t)}{\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}} l_1(t)\int_t^\infty f(s,t)\mathrm{d}s\right).
\end{aligned} \tag{5.39}$$

Substitute (5.29) and (5.39) in HJB equation from Theorem 5.4, to obtain

$$\begin{aligned}
0 &= \tilde{V}_t(t,x) + \left(r(t)x + I_1(t) + I_2(t)\right)\tilde{V}_x(t,x) - \frac{1}{2}\frac{(\mu(t)-r(t))^2\tilde{V}_x^2(t,x)}{\sigma^2(t)\tilde{V}_{xx}} \\
&+ \frac{1-\gamma}{\gamma}\left(\frac{e^{\delta t}}{1-F_{T_1}(t)}\right)^{\frac{1}{\gamma-1}}\tilde{V}_x^{\frac{\gamma}{\gamma-1}}(t,x)\left(w_1^{\frac{1}{1-\gamma}} + w_2^{\frac{1}{1-\gamma}}\right) \\
&+ \left(h_1\left(x+b_2(t)\right)+h_2\left(x+b_1(t)\right)\right)\tilde{V}_x(t,x) \\
&- \left(\left(\frac{h_1(t)}{\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}} l_2(t)\int_t^\infty f(t,z)\mathrm{d}z + \left(\frac{h_2(t)}{\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}} l_1(t)\int_t^\infty f(s,t)\mathrm{d}s\right) \\
&\times e^{\frac{\delta t}{\gamma-1}}\left(1-\frac{1}{\gamma}\right)\tilde{V}_x^{\frac{\gamma}{\gamma-1}}(t,x).
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &= \tilde{V}_t(t,x) + \left(r(t)x + I_1(t) + I_2(t) + h_1\left(x+b_2(t)\right) + h_2\left(x+b_1(t)\right)\right)\times\tilde{V}_x(t,x) \\
&- \frac{1}{2}\frac{(\mu(t)-r(t))^2\tilde{V}_x^2(t,x)}{\sigma^2(t)\tilde{V}_{xx}} \\
&+ e^{\frac{\delta t}{\gamma-1}}\left(\frac{1}{\gamma}-1\right)\tilde{V}_x^{\frac{\gamma}{\gamma-1}}(t,x)\times\left((1-F_{T_1}(t))^{\frac{1}{1-\gamma}}\left(w_1^{\frac{1}{1-\gamma}} + w_2^{\frac{1}{1-\gamma}}\right)\right. \\
&+ \left.\left(\frac{h_1(t)}{\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}} l_2(t)\int_t^\infty f(t,z)\mathrm{d}z + \left(\frac{h_2(t)}{\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}} l_1(t)\int_t^\infty f(s,t)\mathrm{d}s\right).
\end{aligned}$$

To make this simple consider

$$G(t) = e^{\frac{\delta t}{\gamma-1}} \left((1 - F_{T_1}(t))^{\frac{1}{1-\gamma}} \left(w_1^{\frac{1}{1-\gamma}} + w_2^{\frac{1}{1-\gamma}} \right) + \left(\frac{h_1(t)}{\int_t^\infty f(t,z)\mathrm{d}z} \right)^{\frac{\gamma}{\gamma-1}} l_2(t) \int_t^\infty f(t,z)\mathrm{d}z \right.$$
$$\left. + \left(\frac{h_2(t)}{\int_t^\infty f(s,t)\mathrm{d}s} \right)^{\frac{\gamma}{\gamma-1}} l_1(t) \int_t^\infty f(s,t)\mathrm{d}s \right).$$

So the HJB equation will become

$$\begin{aligned} 0 &= \tilde{V}_t(t,x) + (r(t)x + I_1(t) + I_2(t) + h_1(x + b_2(t)) + h_2(x + b_1(t))) \times \tilde{V}_x(t,x) \\ &- \frac{1}{2} \frac{(\mu(t) - r(t))^2 \tilde{V}_x^2(t,x)}{\sigma^2(t) \tilde{V}_{xx}(t,x)} + \left(\frac{1}{\gamma} - 1 \right) \tilde{V}_x^{\frac{\gamma}{\gamma-1}}(t,x) G(t), \end{aligned} \tag{5.40}$$

with the terminal condition

$$\tilde{V}(T,x) = (1 - F_{T_1}(T))\, w_4 e^{-\delta T} U(X(T)).$$

Let us compute the partial derivatives $\tilde{V}_t$, $\tilde{V}_x$ and $\tilde{V}_{xx}$ from ansatz function (5.26) to get

$$\begin{aligned} \tilde{V}_t(t,x) &= \frac{A_t(t)}{\gamma}(x + B(t))^\gamma + A(t)B_t(t)(x + B(t))^{\gamma-1}, \\ \tilde{V}_x(t,x) &= A(t)(x + B(t))^{\gamma-1}, \\ \tilde{V}_{xx}(t,x) &= (\gamma - 1)A(t)(x + B(t))^{\gamma-2}. \end{aligned}$$

Substitute the above partial derivatives in equation (5.40) to get that

$$\begin{aligned} 0 &= \frac{A_t(t)}{\gamma}(x + B(t))^\gamma + A(t)B_t(t)(x + B(t))^{\gamma-1} \\ &+ (r(t)x + I_1(t) + I_2(t) + h_1(x + b_2(t)) + h_2(x + b_1(t)))\, A(t)(x + B(t))^{\gamma-1} \\ &- \frac{1}{2} \frac{(\mu(t) - r(t))^2}{\sigma^2(t)} \times \frac{A^2(t)(x + B(t))^{2(\gamma-1)}}{(\gamma - 1)A(t)(x + B(t))^{\gamma-2}} \\ &+ \left(\frac{1}{\gamma} - 1 \right) A^{\frac{\gamma}{\gamma-1}}(t) \times (x + B(t))^\gamma G(t). \end{aligned}$$

That is,

$$\begin{aligned} 0 &= \frac{A_t(t)}{\gamma}(x + B(t))^\gamma + A(t)B_t(t)(x + B(t))^{\gamma-1} \\ &+ (r(t)x + I_1(t) + I_2(t) + h_1(x + b_2(t)) + h_2(x + b_1(t)))\, A(t)(x + B(t))^{\gamma-1} \\ &- \frac{1}{2} \frac{(\mu(t) - r(t))^2}{\sigma^2(t)} \times \frac{A(t)(x + B(t))^\gamma}{\gamma - 1} \\ &+ \left(\frac{1}{\gamma} - 1 \right) A^{\frac{\gamma}{\gamma-1}}(t) \times (x + B(t))^\gamma G(t). \end{aligned}$$

Divide the last equation by $(x + B(t))^\gamma$

$$\begin{aligned}
0 &= \frac{A_t(t)}{\gamma} + \frac{A(t)B_t(t)}{(x+B(t))} \\
&+ (r(t)x + I_1(t) + I_2(t) + h_1(x + b_2(t)) + h_2(x + b_1(t)))\frac{A(t)}{(x+B(t))} \\
&- \frac{1}{2}\frac{(\mu(t)-r(t))^2}{\sigma^2(t)} \times \frac{A(t)}{(\gamma-1)} + \left(\frac{1}{\gamma} - 1\right) A^{\frac{\gamma}{\gamma-1}}(t)G(t).
\end{aligned}$$

Adding and subtracting the terms $\frac{r(t)A(t)B(t)}{x+B(t)}$, $\frac{h_1(t)A(t)B(t)}{x+B(t)}$ and $\frac{h_2(t)A(t)B(t)}{x+B(t)}$ to the above equation, to obtain

$$\begin{aligned}
0 &= \frac{A_t(t)}{\gamma} + \frac{A(t)B_t(t)}{(x+B(t))} \\
&+ \frac{r(t)xA(t)}{(x+B(t))} + \frac{r(t)A(t)B(t)}{(x+B(t))} - \frac{r(t)A(t)B(t)}{(x+B(t))} \\
&+ \frac{h_1xA(t)}{(x+B(t))} + \frac{h_1(t)A(t)B(t)}{(x+B(t))} - \frac{h_1(t)A(t)B(t)}{(x+B(t))} \\
&+ \frac{h_2xA(t)}{(x+B(t))} + \frac{h_2(t)A(t)B(t)}{(x+B(t))} - \frac{h_2(t)A(t)B(t)}{(x+B(t))} \\
&+ \frac{I_1(t)A(t)}{(x+B(t))} + \frac{I_2(t)A(t)}{(x+B(t))} + \frac{h_1b_2(t)A(t)}{(x+B(t))} + \frac{h_2b_1(t)A(t)}{(x+B(t))} \\
&- \frac{1}{2}\frac{(\mu(t)-r(t))^2}{\sigma^2(t)} \times \frac{A(t)}{(\gamma-1)} + \left(\frac{1}{\gamma} - 1\right) A^{\frac{\gamma}{\gamma-1}}(t)G(t).
\end{aligned}$$

Note that the above differential equation can be separated into two independent boundary value problems for A and B as follows

$$\begin{aligned}
0 &= \frac{A_t(t)}{\gamma} + \frac{r(t)A(t)(x+B(t))}{(x+B(t))} + \frac{h_1A(t)(x+B(t))}{(x+B(t)} + \frac{h_2A(t)(x+B(t))}{(x+B(t))} \\
&- \frac{1}{2}\frac{(\mu(t)-r(t))^2}{\sigma^2(t)} \times \frac{A(t)}{(\gamma-1)} + \left(\frac{1}{\gamma} - 1\right) A^{\frac{\gamma}{\gamma-1}}(t)G(t),
\end{aligned} \tag{5.41}$$

and

$$\begin{aligned}
0 &= \frac{A(t)B_t(t)}{(x+B(t))} - \frac{r(t)A(t)B(t)}{(x+B(t))} - \frac{h_1(t)A(t)B(t)}{(x+B(t))} - \frac{h_2(t)A(t)B(t)}{(x+B(t))} \\
&+ \frac{I_1(t)A(t)}{(x+B(t))} + \frac{I_2(t)A(t)}{(x+B(t))} + \frac{h_1b_2(t)A(t)}{(x+B(t))} + \frac{h_2b_1(t)A(t)}{(x+B(t))}.
\end{aligned} \tag{5.42}$$

After multiplying equation (5.41) by γ, and rearrange it we obtain that

$$\begin{aligned}
0 &= A_t(t) + \left(\gamma(r(t) + h_1 + h_2) + \frac{(\mu(t)-r(t))^2\gamma}{2\sigma^2(t)(\gamma-1)}\right) A(t) \\
&+ (1-\gamma)\, A^{\frac{\gamma}{\gamma-1}}(t)G(t).
\end{aligned} \tag{5.43}$$

Let

$$H(t) = \frac{1}{1-\gamma}\left(\frac{1}{2}\frac{(\mu(t)-r(t))^2\gamma}{(\gamma-1)\sigma^2(t)} - \gamma\left(r(t)+h_1(t)+h_2(t)\right)\right).$$

Identity (5.43) becomes

$$\begin{cases} A_t(t) - (1-\gamma)H(t)A(t) + (1-\gamma)A^{\frac{\gamma}{\gamma-1}}(t)G(t) = 0, \\ A_t(T) = \left(1-F_{T_1}(T)\right) w_4 e^{-\delta T}. \end{cases} \tag{5.44}$$

Also multiply equation (5.42) by $\frac{(x+B(t))}{A(t)}$

$$\begin{cases} B_t(t) - \left(r(t)+h_1(t)+h_2(t)\right)B(t) + I_1(t) + I_2(t) + h_1(t)b_2(t) + h_2(t)b_1(t) = 0, \\ B_t(T) = 0. \end{cases} \tag{5.45}$$

To solve equation (5.44) we assume its solution has the form

$$A(t) = D^{1-\gamma}(t). \tag{5.46}$$

Differentiate $A(t)$ in equation (5.46) with respect to time t, we get

$$A_t(t) = (1-\gamma)D^{-\gamma}(t)D_t(t). \tag{5.47}$$

Substitute (5.46) and (5.47) in equation (5.44) to obtain

$$(1-\gamma)D^{-\gamma}(t)D_t(t) - (1-\gamma)H(t)D^{1-\gamma}(t) + (1-\gamma)D^{-\gamma}(t)G(t) = 0.$$

The above equation can be rewritten as

$$\begin{aligned} &D_t(t) - H(t)D(t) + G(t) = 0, \\ &D(T) = \left(\left(1-F_{T_1}(T)\right) w_4 e^{-\delta T}\right)^{\frac{1}{1-\gamma}}. \end{aligned} \tag{5.48}$$

Equation (5.48) is linear 1^{st} order ODE, so we can solve it explicitly by using the integrating factor method to get

$$D(t) = e^{-\int_t^T H(s)ds}\left(\int_t^T G(s)e^{-\int_t^s H(z)dz}ds + c_1\right).$$

Using the condition $D(T) = \left(\left(1-F_{T_1}(T)\right) w_4 e^{-\delta T}\right)^{\frac{1}{1-\gamma}}$ we obtain

$$D(t) = \left(\left(1-F_{T_1}(T)\right) w_4 e^{-\delta T}\right)^{\frac{1}{1-\gamma}} e^{-\int_t^T H(s)\mathrm{d}s} + \int_t^T e^{-\int_t^s H(z)\mathrm{d}z}G(s)\mathrm{d}s.$$

Substitute the value of $D(t)$ in (5.46) we get

$$A(t) = \left(\left((1 - F_{T_1}(T))\, w_4 e^{-\delta T} \right)^{\frac{1}{1-\gamma}} e^{-\int_t^T H(s)\mathrm{d}s} + \int_t^T e^{-\int_t^s H(z)\mathrm{d}z} G(s)\mathrm{d}s \right)^{1-\gamma}.$$

To find a solution for the boundary value problem (5.45), it is again 1^{st} order linear ODE and it can be solved also by the integrating factor method as

$$\begin{aligned} B(t) &= \left(\int_t^T e^{\int_T^s (r(z)+h_1(z)+h_2(z))\mathrm{d}z} \left(I_1(s) + I_2(s) + h_1(s)b_2(s) + h_2(s)b_1(s)\right) \mathrm{d}s + c_2 \right) \\ &\times \quad e^{\int_T^t (r(z)+h_1(z)+h_2(z))\mathrm{d}z}. \end{aligned}$$

Since $B(T) = 0$, we get $c_2 = 0$. Consequently,

$$B(t) = \int_t^T e^{-\int_t^s [r(z)+h_1(z)+h_2(z)]\mathrm{d}z} \left(I_1(s) + I_2(s) + h_1(s)b_2(s) + h_2(s)b_1(s)\right) \mathrm{d}s.$$

Substitute the value of function $\tilde{V}_x(t,x)$ in equation (5.29) to get

$$\bar{c}_i^*(t,x) = \left(\frac{e^{\delta t} A(t)(x + B(t))^{\gamma-1}}{(1 - F_{T_1}(t))\, w_i} \right)^{\frac{1}{\gamma-1}}, \quad i = 1,2.$$

Substitute the value of $A(t)$ from equation (5.46) to get

$$\bar{c}_i^*(t,x) = \left(\frac{e^{\delta t}}{(1 - F_{T_1}(t))\, w_i} \right)^{\frac{1}{\gamma-1}} \frac{1}{D(t)} (x + B(t)), \quad i = 1,2.$$

Substitute the value of $\tilde{V}_x(t,x)$ and $\tilde{V}_{xx}(t,x)$ in equation (5.29) we get

$$\bar{u}^*(t,x) = -\frac{(\mu(t) - r(t))A(t)(x + B(t))^{\gamma-1}}{\sigma^2(t)(\gamma - 1)A(t)(x + B(t))^{\gamma-2}}.$$

Hence,

$$\bar{u}^*(t,x) = -\frac{\mu(t) - r(t)}{(\gamma - 1)\sigma^2(t)} (x + B(t)).$$

Similarly, substitute the value of $\tilde{V}_x(t,x)$ in equation (5.38) to get

$$\bar{q}_1^*(t,x) = -h_1(t)\,(x + b_2(t)) + h_1(t) \left(\frac{e^{\delta t} A(t)(x + B(t))^{\gamma-1}}{\int_t^\infty f(t,z)\mathrm{d}z} \frac{h_1(t)}{l_2^{1-\gamma}(t)} \right)^{\frac{1}{\gamma-1}}.$$

Thus,

$$\bar{q}_1^*(t,x) = h_1(t) \Bigg(\left(\frac{e^{\delta t} h_1(t)}{\int_t^\infty f(t,z)dz} \right)^{\frac{1}{\gamma-1}} \times \frac{l_2(t)}{D(t)} (x + B(t)) - x - b_2(t) \Bigg).$$

Finally, we do the same for $\bar{q}_2^*(x,t)$ to get

$$\bar{q}_2^*(t,x) = h_2(t)\Bigg(\left(\frac{e^{\delta t}h_2(t)}{\int_t^\infty f(s,t)ds}\right)^{\frac{1}{\gamma-1}} \times \frac{l_1(t)}{D(t)}(x+B(t)) - x - b_1(t)\Bigg).$$

This completes the proof. $\square$

From the Proposition 5.2, we conclude the following observations

- There is a relation between death times of the two wage-earners and optimal strategies.
- We can assume that the human capital of the two wage-earners is $B(t)$, and that is mean their incomes affects on their decisions.
- The consumption and investment are increasing with respect to the human capital of the two wage-earners.
- If the wife's income higher than husband's income, the two wage-earners would be more motivated to buy Welfare for the wife.

5.4.2 CASE 2: No Welfare contracts

In the next proposition, we will do the same thing in previous Proposition 5.2, but when $\bar{q}_1 = \bar{q}_2 = 0$, which means the model includes only life-insurance, consumption and investment in the financial market. In this case the results will as follows.

Proposition 5.3. *Assume $\bar{q}_1 = \bar{q}_2 = 0$, $U(\cdot)$ is given as in (5.25), the conditions of Lemma 5.2 hold, and the value function $V(t,x)$ is as given in (5.27).*
Then the optimal strategies are given by

$$\begin{cases} \bar{c}_i(t,x) = \left(\frac{e^{\delta t}}{(1-F_{T_1}(t))w_i}\right)^{\frac{1}{\gamma-1}} \frac{1}{D(t)}(x+B(t)), i=1,2, \\ \bar{u}(t,x) = -\frac{\mu(t)-r(t)}{(\gamma-1)\sigma^2(t)}(x+B(t)), \\ \bar{k}_1(t,x) = \eta_1(t)\Bigg(\left(\frac{e^{\delta t}\eta_1(t)}{\int_t^\infty f(t,z)dz}\right)^{\frac{1}{\gamma-1}} \times \frac{l_2(t)}{D(t)}(x+B(t)) - x - b_2(t)\Bigg), \\ \bar{k}_2(t,x) = \eta_2(t)\Bigg(\left(\frac{e^{\delta t}\eta_2(t)}{\int_t^\infty f(s,t)ds}\right)^{\frac{1}{\gamma-1}} \times \frac{l_1(t)}{D(t)}(x+B(t)) - x - b_1(t)\Bigg), \end{cases}$$

where

$$D(t) = \Big((1 - F_{T_1}(T))\, w_4 e^{-\delta T}\Big)^{\frac{1}{1-\gamma}} e^{-\int_t^T H(s)\mathrm{d}s} + \int_t^T e^{-\int_t^s H(z)\mathrm{d}z} G(s)\mathrm{d}s,$$

$$B(t) = \int_t^T e^{-\int_t^s (r(z)+\eta_1(z)+\eta_2(z))\mathrm{d}z} \left(I_1(s) + I_2(s) + \eta_1(s)b_2(s) + \eta_2(s)b_1(s)\right)\mathrm{d}s,$$

$$H(t) = \frac{1}{1-\gamma}\left(\frac{1}{2}\frac{(\mu(t)-r(t))^2\gamma}{(\gamma-1)\sigma^2(t)} - \gamma\left(r(t)+\eta_1(t)+\eta_2(t)\right)\right),$$

$$G(t) = e^{\frac{\delta t}{\gamma-1}}\left((1-F_{T_1}(t))^{\frac{1}{1-\gamma}}\left(w_1^{\frac{1}{1-\gamma}} + w_2^{\frac{1}{1-\gamma}}\right) + \left(\frac{\eta_1(t)}{\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}} l_2(t)\int_t^\infty f(t,z)\mathrm{d}z \right.$$
$$\left. + \left(\frac{\eta_2(t)}{\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}} l_1(t)\int_t^\infty f(s,t)\mathrm{d}s\right).$$

Proof. Similar to proof of Proposition 5.2. ◻

5.4.3 CASE 3: Life-insurance is not being control variable

In this section, we assume life - insurance parameters are not control variables in order to find an explicit solution for the other controls (welfare, consumption and investment in the financial market).

The following proposition show an explicit solution when life-insurance parameters are not control variables.

Proposition 5.4. *Assume $\bar{k}_1, \bar{k}_2$ are not control variables, $U(\cdot)$ is as given in (5.25), the conditions of Lemma 5.2 hold, and the value function $V(t,x)$ is as given in (5.27). Then the optimal strategies are given by*

$$\begin{cases} \bar{c}_i(t,x) = \left(\frac{e^{\delta t}}{(1-F_{T_1}(t))w_i}\right)^{\frac{1}{\gamma-1}} \frac{1}{D(t)}(x+B(t)), i = 1,2, \\ \bar{u}(t,x) = -\frac{\mu(t)-r(t)}{(\gamma-1)\sigma^2(t)}(x+B(t)), \\ \bar{q}_1(t,x) = h_1(t)\left(\left(\frac{e^{\delta t}h_1(t)}{\int_t^\infty f(t,z)dz}\right)^{\frac{1}{\gamma-1}} \times \frac{l_2(t)}{D(t)}(x+B(t)) - x - \frac{\bar{k}_1}{\eta_1(t)} - b_2(t)\right), \\ \bar{q}_2(t,x) = h_2(t)\left(\left(\frac{e^{\delta t}h_2(t)}{\int_t^\infty f(s,t)ds}\right)^{\frac{1}{\gamma-1}} \times \frac{l_1(t)}{D(t)}(x+B(t)) - x - \frac{\bar{k}_2}{\eta_2(t)} - b_1(t)\right), \end{cases}$$

where

$$D(t) = \left((1 - F_{T_1}(T))\, w_4 e^{-\delta T}\right)^{\frac{1}{1-\gamma}} e^{-\int_t^T H(s)\mathrm{d}s} + \int_t^T e^{-\int_t^s H(z)\mathrm{d}z} G(s)\mathrm{d}s,$$

$$B(t) = \int_t^T e^{-\int_t^s (r(z)+h_1(z)+h_2(z))\mathrm{d}z}$$
$$\times \left(I_1(s) + I_2(s) - \bar{k}_1(s) - \bar{k}_2(s) + h_1(s)(b_2(s) + \bar{k}_1(s)) + h_2(s)(b_1(s) + \bar{q}_2(s))\right) \mathrm{d}s,$$

$$H(t) = \frac{1}{1-\gamma}\left(\frac{1}{2}\frac{(\mu(t) - r(t))^2\gamma}{(\gamma - 1)\sigma^2(t)} - \gamma\left(r(t) + h_1(t) + h_2(t)\right)\right),$$

$$G(t) = e^{\frac{\delta t}{\gamma-1}}\left((1 - F_{T_1}(t))^{\frac{1}{1-\gamma}}\left(w_1^{\frac{1}{1-\gamma}} + w_2^{\frac{1}{1-\gamma}}\right) + \left(\frac{h_1(t)}{\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}} l_2(t)\int_t^\infty f(t,z)\mathrm{d}z\right.$$
$$\left. + \left(\frac{h_2(t)}{\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}} l_1(t)\int_t^\infty f(s,t)\mathrm{d}s\right).$$

Proof. The proof of this proposition is closely to the technique introduced in proof of Proposition 5.2, by adding the corresponding updates that fit with our model.

For instance, to obtain an optimal welfare purchase strategies when $\bar{k}_1, \bar{k}_2$ are not control variables, we can consider the bivariate function ϕ for $\bar{q}_1, \bar{q}_2$ as

$$\begin{aligned}\phi\left(\bar{q}_1, \bar{q}_2\right) = &-\left(\bar{q}_1 + \bar{q}_2\right)\tilde{V}_x(t,x)\\ &+ V_1\left(t, x + \frac{\bar{k}_2}{\eta_2(t)} + \frac{\bar{q}_2}{h_2(t)}\right) \times \int_t^\infty f(s,t)\mathrm{d}s\\ &+ V_2\left(t, x + \frac{\bar{k}_1}{\eta_1(t)} + \frac{\bar{q}_1}{h_1(t)}\right) \times \int_t^\infty f(t,z)\mathrm{d}z.\end{aligned}$$

And we do the same as we did in Proposition 5.2. □

5.4.4 CASE 4: Welfare policy is not being control variable

The next proposition is similar to the Proposition 5.4, but we assume the welfare parameters are not being control variables. In this case the results will be as follows.

Proposition 5.5. *Assume $\bar{q}_1, \bar{q}_2$ are not control variables, $U(\cdot)$ is as given in (5.25), the conditions of Lemma 5.2 hold, and the value function $V(t,x)$ is as given in (5.27). Then the optimal strategies are given by*

$$
\begin{cases}
\bar{c}_i(t,x) = \left(\frac{e^{\delta t}}{(1-F_{T_1}(t))w_i}\right)^{\frac{1}{\gamma-1}} \frac{1}{D(t)}(x+B(t)), i=1,2,\\
\bar{u}(t,x) = -\frac{\mu(t)-r(t)}{(\gamma-1)\sigma^2(t)}(x+B(t)),\\
\bar{k}_1(t,x) = \eta_1(t)\Bigg(\left(\frac{e^{\delta t}\eta_1(t)}{\int_t^\infty f(t,z)dz}\right)^{\frac{1}{\gamma-1}} \times \frac{l_2(t)}{D(t)}(x+B(t)) - x - \frac{\bar{q}_1}{h_1(t)} - b_2(t)\Bigg),\\
\bar{k}_2(t,x) = \eta_2(t)\Bigg(\left(\frac{e^{\delta t}\eta_2(t)}{\int_t^\infty f(s,t)ds}\right)^{\frac{1}{\gamma-1}} \times \frac{l_1(t)}{D(t)}(x+B(t)) - x - \frac{\bar{q}_2}{h_2(t)} - b_1(t)\Bigg),
\end{cases}
$$

where

$$
\begin{aligned}
D(t) &= \left((1-F_{T_1}(T))\, w_4 e^{-\delta T}\right)^{\frac{1}{1-\gamma}} e^{-\int_t^T H(s)\mathrm{d}s} + \int_t^T e^{-\int_t^s H(z)\mathrm{d}z} G(s)\mathrm{d}s,\\
B(t) &= \int_t^T e^{-\int_t^s (r(z)+\eta_1(z)+\eta_2(z))\mathrm{d}z}\\
&\quad \times \left(I_1(s)+I_2(s)-\bar{q}_1(s)-\bar{q}_2(s)+\eta_1(s)(b_2(s)+\bar{q}_1(s))+\eta_2(s)(b_1(s)+\bar{q}_2(s))\right)\mathrm{d}s,\\
H(t) &= \frac{1}{1-\gamma}\left(\frac{1}{2}\frac{(\mu(t)-r(t))^2\gamma}{(\gamma-1)\sigma^2(t)} - \gamma\left(r(t)+\eta_1(t)+\eta_2(t)\right)\right),\\
G(t) &= e^{\frac{\delta t}{\gamma-1}}\Bigg((1-F_{T_1}(t))^{\frac{1}{1-\gamma}}\left(w_1^{\frac{1}{1-\gamma}}+w_2^{\frac{1}{1-\gamma}}\right) + \left(\frac{\eta_1(t)}{\int_t^\infty f(t,z)\mathrm{d}z}\right)^{\frac{\gamma}{\gamma-1}} l_2(t)\int_t^\infty f(t,z)\mathrm{d}z\\
&\quad + \left(\frac{\eta_2(t)}{\int_t^\infty f(s,t)\mathrm{d}s}\right)^{\frac{\gamma}{\gamma-1}} l_1(t)\int_t^\infty f(s,t)\mathrm{d}s\Bigg).
\end{aligned}
$$

Proof. Similar to the proof of Proposition 5.4. □

Chapter 6

Conclusion

We have extended the work done by Wei et al [45] by allowing a two wage-earners to contribute in the social security system in order to protect their families from risk in the future.

We have studied an optimal control problem after the first death in Chapter 4, first we have transformed the stochastic optimal control problem under consideration of the two wage-earners to an equivalent one with fixed planning horizon, after that we have derived a dynamic programming principle DPP and the corresponding the HJB equation. We have characterized the optimal strategies, after the death of one wage-earner concerning consumption, investment, life-insurance and social welfare policy using power utility functions.

In Chapter 5, we have studied the optimal control problem before the first death when the two wage-earners are contributing in the social security system while participating in the life-insurance markets. We have used Copula model as stochastic mortality model for dependent lives, to handle the stochastic optimal control problems under consideration. First, we have transformed the stochastic optimal control problem of the two wage-earners in which neither of them dies before the retirement date into a one with a fixed planning horizon using the tower rules of conditional expectations. After that we derived a DPP and subsequently we derived the corresponding HJB equation modeling the problem with all controls before first death. Using power utility functions, we have characterized the optimal strategies before the first death, in the following cases (no life-insurance contracts, no welfare policy contracts, life-insurance is not being control variable and welfare policy is not being control variable). Under some conditions, we have determined an explicit solutions for the optimal strategies in each case.

One possible case could be considered for a future work is when the life-insurance and welfare parameters are all control variables before the first death. In this case, it will be hard to derive an explicit solution analytically but numerical solution would be interesting.

Bibliography

[1] S. Ahmad and A. Ambrosetti. A textbook on ordinary differential equations. *Springer Cham Heidelberg*, 2015.

[2] E. Bayraktar and V. R. Young. Life insurance purchasing to maximize utility of household consumption. *N. Am. Actuar. J.*, 17 (2):114–135, 2013.

[3] K. Bruhn and M. Steffensen. Household consumption, investment and life insurance. *Insu: Math. Econ.*, 48:315–325, 2011.

[4] V. Capasso and D. Bakstein. *An introduction to continuous-time stochastic processes: Theory, Models, and Applications to Finance, Biology, and Medicine.* Birkhäuser, 4th edition, 2021.

[5] D. Clayton. A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *O. U. P*, 65 (1):141–151, 1978.

[6] M. Elias and S. Shakarchi. *Measure theory, integration, and hilbert spaces.* New Jersey: Princeton University, 1st edition, 2005.

[7] B. ∅ksendal. *Stochastic Differential Equations: An Introduction with Applications.* New York: Springer-Verlag Heidelberg, 5th edition, 2000.

[8] L. C. Evans. *An Introduction to Stochastic Differential Equations.* Am. Math. Soc., 2012.

[9] M. Evans and J. Rosenthal. *Probability and Statistics: The Science of Uncertainty.* W. H. Freeman, 2nd edition, 2009.

[10] G. Franciszek. Nonhomogeneous compound poisson process application to modeling of random processes related to accidents in the baltic sea waters and ports. *Journal of Polish Safety and Reliability Association*, 9 (3), 2018.

[11] E. W. Frees, J. Carriere, and E. Valdez. Annuity valuation with dependent mortality. *J. Risk. Insur*, 63:229–261, 1996.

[12] C. Genest. Frank's family of bivariate distributions. *O. U. P*, 74 (3):549–555, 1987.

[13] S. Ghalem, B. Kechar, A. Bounceur, and R. Euler. A probabilistic multivariate copula-based technique for faulty node diagnosis in wireless sensor networks. *J. Netw. Comput. Appl.*, 127:9–25, 2019.

[14] A. Gut. Probability: a graduate course. *New York: Springer*, 2005.

[15] N. Hakansson. Optimal investment and consumption strategies under risk, an uncertain lifetime, and insurance. *Int. Econ. Rev.*, 10 (3):443–466, 1969.

[16] N. Han and M. Hung. Optimal consumption, portfolio, and life insurance policies under interest rate and inflation risks. *Insu: Math. Econ.*, 73:4–67, 2017.

[17] J. Harshbarger and J. Reynolds. *Mathematical Applications for the Management, Life, and Social Sciences*. Brooks/Cole Publishing, 10th edition, 2012.

[18] M. Hoshiea, A. S. Mousa, and A. A. Pinto. Optimal social welfare policy within financial and life-insurance markets. *Optimization: A Journal of Mathematical Programming and Operations Research*, 2022.

[19] P. Hougaard. A class of multivariate failure time distributions. *O. U. P*, 73 (3):671–678, 1986.

[20] V. Ivrii. *Partial Differential Equations*. Department of Mathematics: University of Toronto, 2017.

[21] M. Ji, M. Hardy, and J. Li. Markovian approaches to joint-life mortality. *N. Am. Actuar. J.*, 15 (3):357–376, 2011.

[22] H. Kraft and M. Steffensen. Optimal consumption and insurance: a continuous-time markov chain approach. *ASTIN Bull.*, 28:231–257, 2008.

[23] M. Kwak and B. Lim. Optimal portfolio selection with life insurance under inflation risk. *J. Bank Finance*, 46:59–71, 2014.

[24] M. Kwak, Y. Shin, and U. Choi. Optimal portfolio, consumption and retirement decision under a preference change. *J. Math. Anal. Appl.*, 355:527–540, 2009.

[25] M. Kwak, Y. H. Shin, and U. J. Choi. Optimal investment and consumption decision of a family with life insurance. *Insu: Math. Econ.*, 48:176–188, 2011.

[26] A. Lenart. The moments of the gompertz distribution and maximum likelihood estimation of its parameters. *Scand. Actuar. J.*, 2014 (3):255–277, 2012.

[27] E. Luciano, J. Spreeuw, and E. Vigna. Modelling stochastic mortality for dependent lives. *Insu: Math. Econ.*, 43 (2):234–244, 2008.

[28] A. W. Marshall and I. Olkin. A multivariate exponential distribution. *J. Am. Stat. Assoc.*, 62 (317):30–44, 1967.

[29] R. Merton. Lifetime portfolio selection under uncertainty: The continuous time case. *Rev. Econ. Stat*, 51 (3):247–257, 1969.

[30] R. Merton. Optimum consumption and portfolio rules in a continuous-time model. *J. Econ. Theory*, 3 (4):373–413, 1971.

[31] T. Missov, A. Lenart, L. Nemeth, V. Canudas-Romo, and J. Vaupel. The gompertz force of mortality in terms of the modal age at death. *Demogr. Res*, 32:1031–1048, 2015.

[32] K. S. Moore and V. R. Young. Optimal insurance in a continuous-time model. *Insu: Math. Econ.*, 39:47–68, 2006.

[33] A. S. Mousa, D. Pinheiro, S. Pinheiro, and A. A. Pinto. Optimal consumption, investment and life-insurance purchase under a stochastically fluctuating economy. *Optimization: A Journal of Mathematical Programming and Operations Research*, 2022.

[34] A. S. Mousa, D. Pinheiro, and A. A. Pinto. A consumption-investment problem with a diminishing basket of goods. *Operational Research: IO 2013 - XVI Congress of APDIO, (eds. J. P. Almeida, J. F. Oliveira and A. A. Pinto). CIM Series in Mathematical Sciences, Springer*, Chapter 17:295–310, 2015.

[35] A. S. Mousa, D. Pinheiro, and A. A. Pinto. Optimal life-insurance selection and purchase within a market of several life-insurance providers. *Insu: Math. Econ.*, 67:133–141, 2016.

[36] P. Olofsson and M. Andersson. *Probability, Statistics, and Stochastic Processes.* A Wiley-Interscience Publication John Wiley Sons, INC., 2nd edition, 2012.

[37] M. J. Panik. *Stochastic Differential Equations: An Introduction with Applications in Population Dynamics Modeling.* A Wiley-Interscience Publication John Wiley Sons, INC., 1st edition, 2017.

[38] S. Park and B. G. Jang. Optimal retirement strategy with a negative wealth constraint. *Oper. Res. Lett.*, 42 (3):208–212, 2014.

[39] M. Peres, J. Achcar, and E. Martinez. Bivariate lifetime models in presence of cure fraction : a comparative study with many different copula functions. *Heliyon*, 6(6):e03961, 2020.

[40] T. A. Pirvu and H. Zhang. Optimal investment, consumption and life insurance under mean-reverting returns: the complete market solution. *Insu: Math. Econ.*, 51:303–309, 2012.

[41] S. Richard. Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. *J. Financ. Econ*, 2:187–203, 1975.

[42] S. Sarkka and A. Solin. *Applied Stochastic Differential Equations.* Cambridge University Press, 2nd edition, 2019.

[43] G. Thomas, D. Maurice, J. Hass, and R. Hass. *Thomas' calculus.* Pearson Education: Addison-Wesley, 11th edition, 2005.

[44] A. Tolver. *An Introduction to Markov Chains.* Department of Mathematical Sciences, University of Copenhagen, 1st edition, 2016.

[45] J. Wei, X. Cheng, Z. Jin, and H. Wang. Optimal consumption–investment and life-insurance purchase strategy for couples with correlated lifetimes. *Insu: Math. Econ.*, 91:244–256, 2020.

[46] M. Yaari. Uncertain lifetime, life insurance and the theory of the consumer. *Rev. Econ. Stud.*, 32:137–150, 1956.

[47] J. Ye. Optimal life insurance purchase, consumption and portfolio under uncertainty: martingale methods. *In: Proceedings of 2007 American Control Conference*, page 1103–1109, 2007.

[48] X. Zeng and T. S. Durrani. Estimation of mutual information using copula density function. *Electron. Lett.*, 47(8):493–494, 2011.